Topics in Singular Perturbations[1]

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Introduction

The area of singular perturbations is a field of increasing interest to applied mathematicians and one without adequate exposition. The only authoritative treatment to date is a single chapter in the highly recommended "Asymptotic Solutions for Ordinary Differential Equations" by Professor Wolfgang Wasow, a substantial contributor to the study of singular perturbations for over twenty years. Otherwise, one must search through the extensive literature.

This article is a slightly revised version of lecture notes prepared and delivered at Bell Telephone Laboratories, Inc., Whippany, New Jersey, during the summer of 1966. The content is selected and strongly reflects the interests, readings, and research of the author. These lecture notes were an attempt to emphasize mathematically sound, constructive techniques for obtaining asymptotic solutions for representative problems. It was hoped that they would, simultaneously, be a satisfactory introduction to the rapidly growing research literature and that they would prove

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useful to those who encounter asymptotic phenomena in physical situations. The favorable response of many readers has been encouraging, but also indicative of the need for such a survey.

Section 1 introduces singular perturbation problems through examples and includes a formal application of the method of matched asymptotic expansions. Section 2 proceeds much more carefully to consider the asymptotic solution of certain boundary value problems for ordinary differential equations with a small parameter multiplying the highest derivatives. Methods introduced there are generalized in chapters three and four to include singular perturbations of eigenvalue problems and of boundary value problems involving two parameters which simultaneously approach zero. The eigenvalue problems generalize the classical study of the vibrations of a string, while the two-parameter problems presented can, in turn, be generalized to problems involving many parameters. The asymptotic solutions of these problems all feature the phenomenon of "loss of boundary conditions."

In Section 5, a different type of singular perturbation phenomena known as relaxation oscillations is encountered, while in Section 6 a more general discussion concerning perturbations of discontinuous solutions is presented. In particular, expansions are obtained for the amplitude and period of the relaxation oscillations of van der Pol's equation. Section 6 also contains Vasil'eva's method for obtaining the asymptotic solution of initial-value problems for systems of nonlinear equations, including detailed calculations for illustrative examples. Lastly, several representative problems for partial differential equations are presented in Section 7. They include examples where the reduced equation is of lower order than the full equation and examples where the reduced equation is of different type, as well as a problem for Oseen flow. Throughout these sections reference is frequently made to relevant literature cited in a concluding bibliography which contains more than one hundred items.

The mathematical and editorial assistance of Dr. J. A. Cochran in this presentation has been exceedingly valuable and is gratefully acknowledged. The assistance of many other friends and colleagues and of Bell Telephone Laboratories is greatly appreciated. Specifically, allow me to mention only Professors G. E. Latta, J. B. Keller, and W. R. Wasow who taught me about singular perturbations and asymptotic methods, and Professor R. W. McKelvey who asked me to include this article in this collection.

1. Regular and Singular Perturbation Problems

Consider a boundary value problem P_ϵ, depending on a small parameter ϵ. Assume that, as $\epsilon \to 0$, the differential equation and the boundary conditions defining P_ϵ approach limiting forms and define a "limiting boundary value problem" P_0. The solution y_ϵ of P_ϵ depends on ϵ, and, under certain conditions, y_ϵ and (some of) its derivatives will approach limits, usually uniformly, as $\epsilon \to 0$ and $\lim y_\epsilon$ will solve the limiting problem P_0. Moreover, if P_ϵ depends on ϵ analytically, we expect that y_ϵ will also depend analytically on ϵ so that y_ϵ can be constructed by the familiar "method of perturbation" as a power series in ϵ. Typically, the solution is expressed as a formal power series in ϵ, the series is substituted into the differential equation and the expressions for the boundary conditions, coefficients of corresponding powers of ϵ are equated, and—with luck—the resulting equations are solvable successively yielding an expansion which converges to the solution, or at least to an asymptotic expansion of the solution ("asymptotic solution") of the boundary value problem as $\epsilon \to 0$. As an example, consider the boundary value problem

$$\frac{d^2u}{dx^2} + (a(x) + \epsilon a_1(x))\frac{du}{dx} + (b(x) + \epsilon b_1(x))\,u = 0,$$

$$u(0) = c_1, \qquad u'(0) = c_2$$

on the interval $x \in [0, 1]$ where the coefficients are smooth and c_1 and c_2 do not depend on ϵ. [Here, and throughout this paper, all functions introduced in defining boundary value problems are assumed to be "sufficiently differentiable." Less differentiability, in general, implies that the asymptotic expansions obtained must be terminated after a finite number of terms. Further, such functions will be considered independent of ϵ, unless otherwise noted.] This problem can be solved by setting

$$u = u_0(x) + \epsilon u_1(x) + \epsilon^2 u_2(x) + \cdots,$$

where, e.g., we ask that u_0 satisfy the limiting boundary value problem

$$\begin{aligned} u_0'' + a(x)\,u_0' + b(x)\,u_0 &= 0,\\ u_0(0) &= c_1,\\ u_0'(0) &= c_2. \end{aligned}$$

Such problems are known as regular (or ordinary) perturbation problems. For further discussion, see Bellman [1].

Throughout the following, we shall be concerned with (irregular or) singular perturbation problems where $\lim y_\epsilon$, if it exists, will not be attained uniformly. This occurs most frequently for boundary value problems in which the order of the differential equation drops, or its type changes, as $\epsilon \to 0$ so that the boundary conditions for P_ϵ are not appropriate when $\epsilon = 0$ and it is not obvious how P_0 should be defined. Irregular perturbation problems need not be of this type, however, e.g.,

$$y'' - \epsilon^2 y = 0,$$
$$y(0) = 1,$$
$$y'(\infty) = 0.$$

For each fixed $\epsilon > 0$, the solution is $y = e^{-\epsilon x}$, but

$$\lim_{\epsilon \to 0^+} e^{-\epsilon x} = \begin{cases} 1 & \text{if} \quad \epsilon x = o(1), \\ e^{-k} & \text{if} \quad \epsilon x = k + o(1), \\ 0 & \text{if} \quad 1/x = o(\epsilon), \end{cases}$$

so that no single limit will be uniformly valid throughout the x-interval $[0, \infty)$. (Note: The Landau order symbols O and o are used throughout the following discussion. The reader should refer to van der Corput [20] for a discussion of these symbols as well as for the definitions of asymptotic expansions and asymptotic equality.)

Generally, singular perturbation problems feature nonanalytic dependence of y_ϵ on ϵ even in cases where P_ϵ depends on ϵ in a very simple manner. This nonanalytic dependence expresses itself frequently in markedly different behavior as ϵ approaches zero through positive and negative values. Since methods and results are completely analogous, however, henceforth we shall consider $\epsilon > 0$.

For a simple singular perturbation problem to analyze, consider

$$(*) \begin{cases} \epsilon y'' + y' + y = 0, \\ y(0) = \alpha, \qquad y(1) = \beta, \end{cases}$$

for $x \in [0, 1]$. Introducing

$$\rho_1 = \frac{-1 + \sqrt{1 - 4\epsilon}}{2\epsilon}$$

and

$$\rho_2 = \frac{-1 - \sqrt{1 - 4\epsilon}}{2\epsilon}$$

(the roots of the characteristic equation $\epsilon\rho^2 + \rho + 1 = 0$), we obtain the exact solution

$$y = \frac{(\beta - \alpha e^{\rho_2})\, e^{\rho_1 x} + (\alpha e^{\rho_1} - \beta)\, e^{\rho_2 x}}{e^{\rho_1} - e^{\rho_2}}$$

for $x \in [0, 1]$. Since

$$\rho_1 = -1 - \epsilon + O(\epsilon^2)$$

and

$$\rho_2 = -1/\epsilon + 1 + \epsilon + O(\epsilon^2),$$

$$y \to \beta e^{1-x} \quad \text{on} \quad 0 < x \leqslant 1$$

as $\epsilon \to 0$ and convergence is uniform in any x-interval $0 < \delta \leqslant x \leqslant 1$. Note that this limit satisfies the "reduced boundary value problem"

$$u' + u = 0,$$

$$u(1) = \beta.$$

Convergence is, however, nonuniform near $x = 0$ unless $\alpha = \beta e$. Neglecting terms which are asymptotically zero,

$$y \sim \beta e^{-(1-x)\rho_1} + (\alpha - \beta e^{-\rho_1})\, e^{\rho_2 x}$$

and, refining our previous estimate,

$$y = \beta e^{(1-x)} + (\alpha - \beta e)\, e^{-x/\epsilon + x} + O(\epsilon),$$

where the order relation O holds uniformly for all $x \in [0, 1]$.

Following Prandtl's idea, we introduce $t = x/\epsilon$ as a stretching transformation to "blow up" the region of quick transition (physically, the "boundary layer") near $x = 0$ (see Prandtl [91]), and proceed heuristically. Letting a dot represent differentiation with respect to t, we write the "boundary layer equation"

$$\ddot{y} + \dot{y} + \epsilon y = 0$$

and attempt a solution of this equation by the regular perturbation technique. Thus, we ask that

$$y = A + Be^{-t} + O(\epsilon)$$

uniformly for all t in $[0, \infty)$. Asking also that y equal α when $t = 0$, we set $B = \alpha - A$. On the other hand, for x a small, but fixed positive number, $t = x/\epsilon \to \infty$ as $\epsilon \to 0$ so that $y \to A$. Moreover, here (i.e., away from the origin), y should behave like βe^{1-x}, the solution of the reduced boundary value problem, and y should approach βe for small values of x. Matching these two "solutions" then, we set $A = \beta e$ and obtain

$$y = \beta e + (\alpha - \beta e)\, e^{-x/\epsilon} + O(\epsilon)$$

in the "boundary layer" while

$$y = \beta e^{1-x} + O(\epsilon)$$

away from $x = 0$.

Such heuristics are the basis of the method of "matched asymptotic expansions" (or "inner and outer expansions") which was developed and applied to many physically significant problems by Kaplun [54], Kaplun [55], Kaplun and Lagerstrom [57], Proudman and Pearson [92], and others [see Van Dyke [105], Erdélyi [25] and [27], Cole and Kevorkian [18], and especially, Kaplun [56] for further references.] We proceed to give a brief description of the formal application of this method to the preceding problem.

Let $y(x, \epsilon)$ be the solution of

$$(*) \begin{cases} \epsilon y'' + y' + y = 0, \\ y(0) = \alpha, \quad y(1) = \beta, \end{cases} \qquad \text{for} \quad x \in [0, 1].$$

Suppose y possesses an asymptotic expansion y^0 as $\epsilon \to 0$ which is of the form

$$y^0(x, \epsilon) = \sum_{n=0}^{\infty} \epsilon^n g_n(x),$$

$$y^0(1, \epsilon) = \beta,$$

and, *suppose* this "outer expansion" is valid on the interval $0 < \delta \leqslant x \leqslant 1$ where both $\delta = o(1)$ and $\epsilon = o(\delta)$ as $\epsilon \to 0$. Introducing the stretching transformation $t = x/\epsilon$, consider the "inner expansion" y^i of the form

$$y^i(t, \epsilon) = \sum_{n=0}^{\infty} \epsilon^n f_n(t),$$

$$y^i(0, \epsilon) = \alpha,$$

and suppose this expansion is valid for $x = \epsilon t = o(1)$ as $\epsilon \to 0$. Then both the inner and outer expansions hold in an "overlap region" [where both $x = o(1)$ and $\epsilon/x = o(1)$] where they can be matched. Expressing y^{i} in terms of the outer variable x and expanding the result for small ϵ and fixed x, we obtain $(y^{\text{i}})^{\text{o}}$, the outer expansion of the inner expansion. The inner expansion of the outer expansion, $(y^{\text{o}})^{\text{i}}$, is analogously defined. Having assumed the existence of an overlap region, the expansions $(y^{\text{i}})^{\text{o}}$ and $(y^{\text{o}})^{\text{i}}$ are related to each other there by the stretching transformation. However as Kaplun and Lagerstrom [57] admit, "there is no *à priori* reason for their regions of validity to overlap." Thus results obtained by formal manipulation should not be *à priori* assumed asymptotically correct. Nevertheless, we proceed formally to demonstrate the procedure.

We find that

$$g_0(x) = \beta e^{(1-x)}$$

and

$$g_1(x) = (1 - x)\, \beta e^{(1-x)},$$

so that the two-term outer expansion is

$$y_2^{\text{o}} = \beta e^{(1-x)}[1 + \epsilon(1 - x)].$$

Writing this in terms of t, expanding for small ϵ, and terminating the result after two terms,

$$\begin{aligned}(y_2^{\text{o}})_2^{\text{i}} &= \beta e[1 - \epsilon t + \epsilon] \\ &= \beta e[1 - x + \epsilon].\end{aligned}$$

Likewise, the two-term inner expansion is of the form

$$y_2^{\text{i}} = [a_0 + (\alpha - a_0)\, e^{-t}] + \epsilon[-(a_0 t + a_1) + e^{-t}((\alpha - a_0)\, t + a_1)]$$

for a_0 and a_1 undetermined constants.

Writing this in terms of x, expanding for small ϵ, and terminating after two terms, yields

$$(y_2^{\text{i}})_2^{\text{o}} = a_0(1 - x) - \epsilon a_1 .$$

Thus, matching $(y_2^{\text{o}})_2^{\text{i}}$, and $(y_2^{\text{i}})_2^{\text{o}}$, we select

$$a_0 = -a_1 = \beta e$$

and the two-term inner and outer solutions are completely determined (formally!).

Knowing the solutions y^o and y^i, we obtain a composite expansion y^c which, hopefully, is uniformly valid for $x \in [0, 1]$. The simplest method is to let

$$y^c \sim \begin{cases} y^i + y^o - (y^o)^i, \\ y^i + y^o - (y^i)^o. \end{cases}$$

Expressing previous results in terms of the outer variable x, we have

$$y_2{}^c = \beta e^{(1-x)}[1 + \epsilon(1 - x)] + e^{-x/\epsilon}[(\alpha - \beta e)(1 + x) - \epsilon\beta e],$$

which should be compared with the known exact solution.

A careful analysis of the method of matched asymptotic expansions has recently been presented by Fraenkel [31]. This paper distinguishes between the idea of overlapping and the "asymptotic matching principle", and analyzes independently the two corresponding techniques for establishing the relationship between the inner and outer expansions.

2. The Asymptotic Solution of Boundary Value Problems for Ordinary Differential Equations Containing a Parameter

Many boundary value problems P_ϵ depend on a small positive parameter ϵ in such a way that the full differential equation is of higher order than the reduced one obtained by setting the small parameter ϵ equal to zero. (For surveys of such problems which have physical significance, see Friedrichs [36], Carrier [10], and Segel [95].) Obvious questions arise. Does the boundary value problem P_ϵ have a limiting solution as $\epsilon \to 0$? If so, (a) does the limiting solution satisfy the reduced equation, and (b) which, if any, of the boundary conditions will be satisfied by the limiting solution? Indeed, when the limiting solution exists and solves the reduced equation (i.e., in cases of "regular degeneration"), we expect the solution of the original problem to, in general, converge nonuniformly near the boundary due to the loss of boundary conditions required in converging to the limiting solution. Such regions of nonuniform convergence are known as boundary layers, in reference to Prandtl's boundary layer theory for viscous flow past a body at high Reynolds number (see Prandtl [91]). Since the limiting solution fails, in general, to be uniformly valid up to the boundary, the straightforward (or regular) perturbation procedure fails and we are faced with a singular perturbation problem.

Wasow [109], following the earlier results of Birkhoff [2], Noaillon [79], and Turrittin [101], stated in a most elegant fashion sufficient conditions for convergence to a limiting solution for homogeneous linear ordinary differential equations depending linearly on a parameter. Latta [63] went further by giving a complete uniformly valid asymptotic expansion for solutions of such equations as well as for some linear partial differential equations. As might be expected, Latta's success was based upon making a change of variables to "blow up" the boundary region in order to accurately describe the nonuniformly converging terms in the solution of the boundary value problem. Cochran [14] formalized this technique by introducing such variables as new independent variables and, by so doing, was able to solve turning point problems and some nonlinear problems as well as those solved by Latta. An extension of these techniques to certain boundary value problems where the orders of differentiation in the boundary conditions may be smaller at $\epsilon = 0$ then for $\epsilon > 0$ has been obtained by O'Malley and Keller [88]. Among other papers discussing these and similar problems Visik and Lyusternik [107] and Harris [46] are especially recommended.

Specifically, let us consider the limiting behavior, as ϵ tends to zero, of solutions of boundary value problems for homogeneous equations of the form

$$\epsilon M\Phi + N\Phi = 0$$

with suitable boundary conditions where $M\Phi$ and $N\Phi$ are linear differential expressions involving one or more independent variables with m, the order of M, greater than n, that of N. Restricting our consideration, for now, to ordinary differential equations on the x-interval [0, 1] without turning points, let

$$My = y^{(m)} + \alpha_1(x)\, y^{(m-1)} + \cdots + \alpha_k(x)\, y^{(m-k)},$$

and

$$Ny = \beta(x)[y^{(n)} + \beta_1(x)\, y^{(n-1)} + \cdots + \beta_j(x)\, y^{(n-j)}]$$

where all coefficients are real and $\beta(x) \neq 0$ for $x \in [0, 1]$.

Further, let us solve

$$\epsilon My + Ny = 0 \tag{2.1}$$

subject to the m boundary conditions

$$\begin{cases} y^{(\lambda_i)}(0) = l_i\,, & i = 1, 2,..., r \\ y^{(\tau_i)}(1) = l_{r+i}\,, & i = 1, 2,..., m - r \end{cases} \tag{2.2}$$

where

$$m > \lambda_1 > \lambda_2 > \cdots > \lambda_r \geqslant 0$$

and

$$m > \tau_1 > \tau_2 > \cdots > \tau_{m-r} \geqslant 0.$$

Before studying the general problem, however, we determine the asymptotic solution of the two-point boundary value problem

$$\begin{cases} \epsilon y'' + a(x)\, y' + b(x)\, y = 0, \\ y(0), \quad y(1) \qquad \text{prescribed}, \end{cases} \tag{2.3}$$

where $a(x)$ is positive for $x \in [0, 1]$.

In studying singular perturbation problems, we continually examine simple illustrative examples—especially constant coefficient problems which can be explicitly integrated. The detailed examination of such problems and their asymptotic solutions is most valuable in gaining direction for studying more general problems. Hence, recall that in Section 1 the solution of the boundary value problem

$$\epsilon y'' + y' + y = 0,$$

$$y(0) = \alpha, \quad y(1) = \beta$$

was found to be

$$y = \frac{(\beta - \alpha e^{\rho_2})\, e^{\rho_1 x} + (\alpha e^{\rho_1} - \beta)\, e^{\rho_2 x}}{e^{\rho_1} - e^{\rho_2}}$$

for $x \in [0, 1]$ where ρ_1 and ρ_2 are the roots of the characteristic equation

$$\epsilon \rho^2 + \rho + 1 = 0.$$

Here, $y \to \beta e^{1-x}$, the solution of the "reduced boundary value problem"

$$u' + u = 0,$$

$$u(1) = \beta,$$

uniformly on any x interval $0 < \delta \leqslant x \leqslant 1$, but, in general, convergence is nonuniform near $x = 0$. Further, since

$$y \sim \beta e^{-(1-x)\rho_1} + (\alpha - \beta e^{-\rho_1})\, e^{\rho_2 x}$$

we see that this nonuniform convergence is due to the exponential decay of $e^{\rho_2 x}$, ρ_2 being that root of the auxiliary polynomial which becomes singular as $\epsilon \to 0$.

After studying this problem where $a(x) = b(x) = 1$, we expect convergence of the solution $y(x)$ of (2.3) on each interval $0 < \delta \leqslant x \leqslant 1$ to

$$u(x) = y(1) \exp\left[\int_x^1 \frac{b(t)}{a(t)}\,dt\right],$$

the solution of the reduced boundary value problem

$$\begin{cases} a(x)\,u' + b(x)\,u = 0, \\ u(1) = y(1). \end{cases}$$

Moreover, we expect nonuniform convergence near $x = 0$ where we expect the boundary layer behavior to be essentially related to the singular root of the auxiliary equation

$$\epsilon D^2 + a(x)\,D + b(x) = 0,$$

i.e.,

$$D_1 = -\frac{a(x)}{2\epsilon}\left(1 + \sqrt{1 - \frac{4\epsilon b(x)}{a^2(x)}}\right)$$

Expanding this root as a function of the small parameter ϵ,

$$\begin{aligned} D_1 &= -\frac{d_1(x, \epsilon)}{\epsilon} = -\frac{1}{\epsilon}\left[a(x) - \frac{\epsilon b(x)}{a(x)} - \frac{\epsilon^2 b^2(x)}{a^3(x)} - \cdots\right] \\ &= -\frac{\tilde{d}_1(x, \epsilon)}{\epsilon} + O(\epsilon), \end{aligned}$$

where

$$\tilde{d}_1(x, \epsilon) \equiv a(x) - \epsilon b(x)/a(x)$$

is positive for ϵ sufficiently small. Again, generalizing from the case when $a(x) = b(x) \equiv 1$, we assume the solution $y(x)$ of (2.3) has the form

$$y(x) \sim A(x, \epsilon) + B(x, \epsilon) \exp\left[-\frac{1}{\epsilon}\int_0^x \tilde{d}_1(s, \epsilon)\,ds\right] \tag{2.4}$$

where

$$A(x, \epsilon) \sim \sum_{r=0}^{\infty} a_r(x)\,\epsilon^r,$$

and

$$B(x, \epsilon) \sim \sum_{r=0}^{\infty} b_r(x) \epsilon^r.$$

Note: (1) The function $1/\epsilon \int_0^x \tilde{d}_1(s, \epsilon)$ is analogous to the stretching transformation used by Latta and the new independent variable introduced by Cochran to investigate the region of nonuniformity.

(2) Since the right member of (2.4) asymptotically equals $a_0 + O(\epsilon)$ in any x-interval $0 < \delta \leqslant x \leqslant 1$, we expect that $a_0(x)$ will equal

$$y(1) \exp\left[\int_x^1 \frac{b(t)}{a(t)} dt\right].$$

(3) Here and below the expansions obtained will not necessarily be asymptotic in the sense of Poincaré, but in the more general sense of van der Corput [20] for the asymptotic sequence $\{\epsilon^n\}$. Note that the coefficients in the power series expansion (2.4) of $y(x, \epsilon)$ depend on ϵ through the exponential factor. As in all such expansions, then, the function $y(x, \epsilon)$ does not uniquely determine the coefficients of its expansion.

Substituting (2.4) into (2.3) and, for each integer r, formally equating to zero the coefficients of ϵ^r and

$$\epsilon^r \exp\left[-\frac{1}{\epsilon}\int_0^x \tilde{d}_1(s, \epsilon)\, ds\right],$$

we obtain

$$a(x)\, a_r'(x) + b(x)\, a_r(x) = -a_{r-1}''(x)$$

and

$$a(x)\, b_r'(x) + a'(x)\, b_r(x) = c_{r-1}(x),$$

where

$$c_{r-1}(x) = b_{r-1}''(x) + 2b_{r-1}'(x)\frac{b(x)}{a(x)} + b_{r-1}(x)\left(\frac{b^2(x)}{a^2(x)} + \left(\frac{b(x)}{a(x)}\right)'\right)$$

with

$$a_{-1}(x) = c_{-1}(x) \equiv 0.$$

Since $y(1) \sim A(1, \epsilon)$, we set $a_0(1) = y(1)$ and $a_k(1) = 0$ for $k \geqslant 1$.

Thus,

$$a_0(x) = y(1) \exp\left[-\int_1^x \frac{b(s)}{a(s)}\, ds\right]$$

and later coefficients are determined successively by the formula

$$a_r(x) = -\int_1^x \frac{a''_{r-1}(t)}{a(t)} \exp\left[-\frac{1}{\epsilon}\int_t^x \frac{b(s)}{a(s)}\, ds\right] dt.$$

Likewise, since $y(0) = A(0, \epsilon) + B(0, \epsilon)$, we set $b_0(0) = y(0) - a_0(0)$ and $b_k(0) = -a_k(0)$ for $k \geqslant 1$. Integrating, the b_r's are determined successively from the formulas

$$b_0(x) = \frac{a(0)}{a(x)} [y(0) - a_0(0)]$$

and

$$b_r(x) = \frac{1}{a(x)} \left[-a_r(0)\, a(0) + \int_0^x c_{r-1}(s)\, ds\right] \qquad \text{for} \quad r \geqslant 1.$$

Thus, to establish the validity of (2.4), we need only prove the following theorem:

Theorem 2.1. *Let $y(x)$ solve the boundary value problem*

$$\epsilon y'' + a(x)\, y' + b(x)\, y = 0,$$

$$y(0), \quad y(1) \qquad \textit{prescribed},$$

where $a(x)$ is positive for $x \in [0, 1]$, and let $a_r(x)$ and $b_r(x)$ be the functions defined successively above.

Further, for each positive integer N, let

$$y(x) = \sum_{r=0}^{N} a_r(x)\, \epsilon^r + \left(\sum_{r=0}^{N} b_r(x)\, \epsilon^r\right) \exp\left[-\frac{1}{\epsilon}\int_0^x \left(a(s) - \epsilon \frac{b(s)}{a(s)}\right) ds\right] + \epsilon^{N+1} R_N(x, \epsilon). \tag{2.5}$$

Then,

$$R_N(x, \epsilon) = O(1) \qquad \textit{for all} \qquad x \in [0, 1].$$

PROOF OF ASYMPTOTIC CONVERGENCE

The reader who is interested only in the construction of formal asymptotic solutions may, without much loss, omit this proof. The serious student of asymptotics, however, will realize that formal procedures require justification. Thus, the stated results will be established in a manner which emphasizes methods of proof which are applicable generally. Results from integral equations theory used may be found, e.g., in Tricomi [100], while a general discussion of the use of integral equations to prove asymptotic convergence will be found in Erdélyi [28].

Before establishing asymptotic convergence we first derive some subsidiary lemmas.

Lemma 1. *Consider the initial-value problem:*

$$\begin{cases} \epsilon R'' + a(x) R' + b(x) R = f(x, \epsilon) \\ R(0) = 0, \quad R'(0) = c/\epsilon \end{cases} \tag{2.6}$$

for $x \in [0, 1]$ *where* $0 < \delta \leqslant a(x)$, $\epsilon > 0$, *and* a, b, f, *and* c *are bounded. Then* (2.6) *has a unique bounded solution* R *for all* $x \in [0, 1]$.

Proof. Integrating the differential equation,

$$R'(x) = \frac{c}{\epsilon} \exp\left[-\frac{1}{\epsilon}\int_0^x a(s)\, ds\right] + \int_0^x \left[\frac{f(t) - b(t) R(t)}{\epsilon}\right] \exp\left[-\frac{1}{\epsilon}\int_t^x a(s)\, ds\right] dt.$$

Integrating again and changing the order of integration:

$$(*) \qquad R(x) = \frac{c}{\epsilon}\int_0^x \exp\left[-\frac{1}{\epsilon}\int_0^z a(s)\, ds\right] dz + \frac{1}{\epsilon}\int_0^x \left(\int_t^x \exp\left[-\frac{1}{\epsilon}\int_t^z a(s)\, ds\right] dz\right) (f(t) - b(t) R(t))\, dt.$$

This integral equation is, however, uniquely solvable by successive approximations. We merely set

$$R_0(x) = \frac{c}{\epsilon}\int_0^x \exp\left[-\frac{1}{\epsilon}\int_0^z a(s)\, ds\right] dz + \frac{1}{\epsilon}\int_0^x \int_t^x \exp\left[-\frac{1}{\epsilon}\int_t^z a(s)\, ds\right] dz\, f(t)\, dt$$

and

$$R_j(x) = R_0(x) - \frac{1}{\epsilon}\int_0^x \int_t^x b(t)\, R_{j-1}(t) \exp\left[-\frac{1}{\epsilon}\int_t^z a(s)\, ds\right] dz\, dt \qquad j \geqslant 1$$

and let $R(x) \equiv \lim_{j\to\infty} R_j(x)$. Convergence follows from the estimates

$$0 \leqslant \int_0^x \exp\left[-\frac{1}{\epsilon}\int_0^z a(s)\, ds\right] dz \leqslant \int_0^x e^{-\delta z/\epsilon}\, dz = \frac{\epsilon}{\delta}\left[1 - e^{-\delta x/\epsilon}\right]$$

and

$$\int_0^x \int_x^t \exp\left[-\frac{1}{\epsilon}\int_t^z a(s)\, ds\right] g(t)\, dz\, dt \leqslant \frac{2\epsilon}{\delta}\int_0^x g(t)\, dt \qquad \text{for} \quad g \text{ positive.}$$

Having established the existence and uniqueness of R, its boundedness also follows from the same estimates. Specifically,

$$|\, R(x)| \leqslant \psi(x) \equiv \frac{|\, c\,|}{\delta}(1 - e^{-\delta x/\epsilon}) + \frac{2M}{\delta}\int_0^x (1 + |\, R(t)|)\, dt$$

where M is a sufficiently large positive constant. (Below, M will represent a generic positive constant, i.e., M will not necessarily be the same constant each time it appears.) This implies that

$$\psi'(x) \leqslant \frac{|\, c\,|}{\epsilon}\, e^{-\delta x/\epsilon} + \frac{2M}{\delta}(1 + \psi(x))$$

and

$$\psi(0) = 0$$

which, in turn, implies that

$$|\, R(x)| \leqslant \psi(x) \leqslant \int_0^x \left(\frac{|\, c\,|}{\epsilon}\, e^{-\delta t/\epsilon} + \frac{2M}{\delta}\right) \cdot e^{(2M/\delta)(x-t)}\, dt,$$

so $R(x)$ is bounded throughout $[0, 1]$.

Note, further, that the solution of (2.6) depends continuously on the value of c. Specifically, if $\tilde{R}(x)$ and $\tilde{\tilde{R}}(x)$ represent solutions of (2.6) with c values $\tilde{c}$ and $\tilde{\tilde{c}}$, respectively, the previous estimates imply that

$$|\, \tilde{R}(x) - \tilde{\tilde{R}}(x)| \leqslant \frac{|\, \tilde{c} - \tilde{\tilde{c}}\,|}{\epsilon}\int_0^x e^{-\delta t/\epsilon}\, e^{(2M/\delta)(x-t)}\, dt.$$

Using these estimates, we easily show that

$$R(x) = c\left[-\frac{1}{a(x)}\exp\left[-\frac{1}{\epsilon}\int_0^x a(s)\,ds\right] + \frac{1}{a(0)}\exp\left[-\int_0^x \frac{b(s)}{a(s)}\,ds\right]\right]$$
$$+ \int_0^x \frac{f(t,\epsilon)}{a(t)}\exp\left[-\int_t^x \frac{b(s)}{a(s)}\,ds\right]dt + O(\epsilon)$$

Since R is unique and depends continuously on c, there is a unique value for c such that the solution R of the initial-value problem (2.6) solves the boundary value problem

$$\epsilon R'' + a(x)\,R' + b(x)\,R = f(x,\epsilon),$$
$$R(0) = R(1) = 0.$$

Clearly, for any bounded solution R, c must satisfy the equation

$$c\int_0^1 \exp\left[-\frac{1}{\epsilon}\int_0^z a(s)\,ds\right]dz = -\int_0^1\int_t^1 (f(t) - b(t)\,R(t))$$
$$\cdot \exp\left[-\frac{1}{\epsilon}\int_t^z a(s)\,ds\right]dz\,dt.$$

Thus, we have proved the existence half of the following statement.

Lemma 2. *For all sufficiently small values of ϵ, there exists a unique bounded solution of the boundary value problem*

$$\begin{cases}\epsilon R'' + a(x)\,R' + b(x)\,R = f(x,\epsilon),\\ R(0) = R(1) = 0,\end{cases} \tag{2.7}$$

where $0 < \delta \leqslant a(x)$, and a, b, and f are bounded for $x \in [0, 1]$.

Proof. To establish uniqueness, we need only show that the homogeneous problem

$$\epsilon R'' + a(x)\,R' + b(x)\,R = 0,$$
$$R(0) = R(1) = 0$$

has only the trivial solution. Integrating this equation and applying the boundary conditions

$$R(x) = \int_0^1 \psi(z,\epsilon)\,R(z)\,dz \int_0^x \exp\left[-\frac{1}{\epsilon}\int_0^t a(s)\,ds\right]dt$$
$$- \int_0^x\int_t^x \frac{b(t)\,R(t)}{\epsilon}\exp\left[-\frac{1}{\epsilon}\int_t^s a(r)\,dr\right]ds\,dt$$

where

$$\psi(z, \epsilon) \equiv \frac{b(z) \int_z^1 \exp\left[-\frac{1}{\epsilon}\int_z^t a(s)\, ds\right] dt}{\epsilon \int_0^1 \exp\left[-\frac{1}{\epsilon}\int_0^t a(s)\, ds\right] dt},$$

or

$$R(x) = \int_0^1 C(x, z, \epsilon)\, R(z)\, dz + \int_0^x B(x, z, \epsilon)\, R(z)\, dz$$

where C and B are defined in the obvious way.

Introducing $H(x, y, \epsilon)$, the resolvent kernel corresponding to the Volterra kernel $B(x, y, \epsilon)$ with eigenparameter $\lambda = 1$, multiply the above equation by $H(t, x, \epsilon)$ and integrate from $x = 0$ to $x = t$. Then, using the Fredholm identity

$$B(t, \tau, \epsilon) + H(t, \tau, \epsilon) = \int_\tau^t H(t, x, \epsilon)\, B(x, \tau, \epsilon)\, dx$$

and the integral equation, we obtain

$$\begin{aligned} R(t) &= \int_0^1 \left[C(t, \tau, \epsilon) + \int_0^t C(x, \tau, \epsilon)\, H(t, x, \epsilon)\, dx\right] R(\tau)\, d\tau \\ &\equiv \varphi(t, \epsilon) \int_0^1 \psi(\tau, \epsilon)\, R(\tau)\, d\tau \end{aligned}$$

where

$$\begin{aligned} \varphi(t, \epsilon) \equiv &\int_0^t \exp\left[-\frac{1}{\epsilon}\int_0^z a(s)\, ds\right] dz \\ &+ \int_0^t \left(\int_0^x \exp\left[-\frac{1}{\epsilon}\int_0^z a(s)\, ds\right] dz\right) H(t, x, \epsilon)\, dx. \end{aligned}$$

This, however, is a Fredholm integral equation (with degenerate kernel) having only the trivial solution $R \equiv 0$ as $\epsilon \to 0$. Q.E.D.

Note. (1) The convenient use of the resolvent kernel in this proof follows an analogous argument in Cochran [14] while the technique is applied to more general two-point boundary value problems in Cochran [15]. A related theorem for analogous nonlinear singular perturbation problems is found in Willett [118].

(2) The preceding theorem follows immediately since R_N satisfies a boundary value problem of the form (2.7).

(3) If $a(x) < 0$, the transformation $z = 1 - x$ will allow the reader to solve the boundary value problem

$$\epsilon y'' + a(x)\, y' + b(x)\, y = 0,$$
$$y(0), \quad y(1) \qquad \text{prescribed}$$

as a boundary value problem of type (2.3) for $z \in [0, 1]$. Clearly, the boundary layer behavior will then occur in the vicinity of $x = 1$.

Higher-Order Problems

Now consider the equation

$$\epsilon My + Ny = 0 \tag{2.1}$$

for $m > n$ and $\beta(x) \neq 0$ for $x \in [0, 1]$ subject to the m boundary conditions

$$\begin{cases} y^{(\lambda_i)}(0) = l_i\,, & i = 1, 2,\ldots, r, \\ y^{(\tau_i)}(1) = l_{r+i}\,, & i = 1, 2,\ldots, m - r. \end{cases} \tag{2.2}$$

Letting $y_l(x, \epsilon)$, $l = 1, 2,\ldots, m$, be a fundamental system of linearly independent solutions of (2.1), any solution $y(x, \epsilon)$ will be a linear combination of the y_l's with coefficients which are independent of x. The linear combination will be a solution of the problem (2.1)–(2.2), if the coefficients are chosen to satisfy the boundary conditions. Moreover, the set of m linear equations resulting will be uniquely solvable provided the determinant of the coefficient matrix is nonvanishing.

We seek a particular fundamental set of solutions of the form

$$y(x, \epsilon) = G(x, \epsilon) \exp\left[\frac{1}{\kappa}\int_t^x \tilde{d}(s, \kappa)\, ds\right]$$

where

$$\kappa = \epsilon^{1/(m-n)}$$

and t is a constant. Differentiating j times with respect to x, then,

$$y^{(j)}(x, \epsilon) = \Big\{G^{(j)} + \frac{j}{\kappa} G^{(j-1)}\tilde{d} + \cdots + \frac{1}{\kappa^{j-1}}\left[jG'\tilde{d}^{j-1} + \frac{j(j-1)}{2} G\tilde{d}^{j-2}\tilde{d}'\right]$$
$$+ \frac{1}{\kappa^j} G\tilde{d}^j\Big\} \exp\left[\frac{1}{\kappa}\int_t^x \tilde{d}(s, \kappa)\, ds\right].$$

Substituting into (2.1) and analyzing the result, we obtain the desired fundamental system. Note that the auxiliary equation

$$\epsilon[D^m + \alpha_1(x)\, D^{m-1} + \cdots + \alpha_k(x)\, D^{m-k}] \\ + \beta(x)[D^n + \beta_1(x)\, D^{n-1} + \cdots + \beta_j(x)\, D^{n-j}] = 0 \tag{2.8}$$

associated with (2.1) has $m - n$ solutions of the form

$$\frac{d(x, \kappa)}{\kappa} = \frac{\tilde{d}(x, \kappa)}{\kappa} + O(\kappa)$$

where

$$\tilde{d}(x, \kappa) = \omega^r h(x) + \kappa \left[\frac{\beta_1(x) - \alpha_1(x)}{m - n}\right]$$

for $h(x)$ positive, r an integer, and ω a complex number of modulus one such that

$$(\omega^r h(x))^{m-n} = -\beta(x).$$

To each of these $m - n$ complex solutions of (2.8), we let correspond a formal complex solution of (2.1), namely,

$$y = \frac{A(x, \kappa)}{(h(x))^{(m+n-1)/2}} \exp\left[\frac{1}{\kappa}\int_t^x \tilde{d}(s, \kappa)\, ds\right], \tag{2.9}$$

where A has a power series expansion in κ whose first term is independent of x and whose higher-order terms may each be successively determined to within an additive constant, and t is either 0 or 1.

Note. Since the roots of (2.8) occur in conjugate complex pairs, it is perhaps more natural to associate two formal real solutions of (2.1) with each conjugate pair of roots of (2.8).

Let p be the number of determinations $\tilde{d}$ for which

$$\operatorname{Re} \tilde{d}(x, \epsilon) \tag{2.10}$$

is negative for ϵ sufficiently small and let q be the number for which it is positive. When

$$p + q = m - n$$

we call (2.10) nonexceptional, while we call it exceptional if

$$p + q < m - n.$$

Note. (1) Here, p is equal to the number of roots of the characteristic equation (2.8) which approach $-\infty$ (throughout [0, 1]) as $\epsilon \to 0$, while q is the number which approach $+\infty$. Moreover, $p + q = m - n - 2$ in the exceptional case.

(2) For more general equations, (2.10) may be replaced by the integral

$$\mathrm{Re} \int_0^1 \tilde{d}(s, \epsilon)\, ds$$

(cf., O'Malley and Keller [88]).

Corresponding to the q distinct roots with (2.10) positive, define q linearly independent solutions $y_1, y_2, \ldots, y_q$ of the above form, all with $t = 1$, and corresponding to the p distinct roots with (2.10) negative, p linearly independent solutions, $y_{m-n-p+1}, \ldots, y_{m-n}$, all with $t = 0$. Clearly, $y_1, y_2, \ldots, y_q$, and their derivatives are all exponentially small away from $x = 1$, while $y_{m-n-p+1}, \ldots, y_{m-n}$ exhibit this boundary layer behavior at $x = 0$. In the exceptional case, we also define solutions $y_{q+1}, y_{q+2}, \ldots, y_{m-n-p}$ by the formula (2.9) but with t, as yet, undetermined. In addition to these $m - n$ "boundary layer type" solutions, n additional linearly independent asymptotic solutions of (2.1) can be obtained as regular perturbations of any complete set of fundamental solutions of the reduced equation $Nz = 0$. We denote them by $y_{m-n+k}(x, \epsilon)$, $k = 1, 2, \ldots, n$, with each solution a series in powers of ϵ for which each variable coefficient is determined up to an additive constant by the differential equation (2.1) and the fundamental solution z_k of $Nz = 0$ which it perturbs. That $y_1, \ldots, y_m$ form a fundamental system for (2.1) follows from the results of Turrittin [102], as outlined in Wasow [116].

We define the reduced boundary value problem to consist of the reduced equation $Nz = 0$ plus n of the boundary conditions (2.2), and state which $m - n$ boundary conditions are omitted in the following

Cancellation Law. (1) *Cancel p boundary conditions at $x = 0$ and q boundary conditions at $x = 1$, starting from those containing the highest derivative.*

(2) *In the exceptional case, also cancel from the boundary conditions those $m - n - p - q$ of highest order of differentiation. These boundary conditions must all belong to the same end point (that end point being denoted by $\tilde{t}$), and their selection must be possible without ambiguity.*

All together, then, let P be the number of boundary conditions canceled at $x = 0$ and let Q be the number canceled at $x = 1$. Having obtained a complete set of formal asymptotic linearly independent solutions of (2.1), we expect to write the limiting solution of the boundary value problem (if it exists) as a linear combination of these solutions. Thus, we are led to setting

$$y(x) = \kappa^{\alpha_1} \sum_{k=1}^{Q} y_k(x, \kappa) + \kappa^{\alpha_2} \sum_{k=Q+1}^{m-n} y_k(x, \kappa) + \sum_{k=1}^{n} y_{m-n+k}(x, \kappa),$$

where the y_k, $k \leqslant m - n$, are the linearly independent solutions defined above, and the $y_{m-n+k}(x, \kappa)$ are multiplies (with coefficients power series in κ with constant coefficients) of the solutions $y_{m-n+k}(x, \epsilon)$ defined above. In the exceptional case, we set $t = \tilde{t}$ in the expressions for $y_{q+1}, y_{q+2}, \ldots, y_{m-n-p}$. α_1 and α_2 are defined in the following manner:

Nonexceptional case:

$$\alpha_1 = \tau_Q, \qquad \alpha_2 = \lambda_P.$$

Exceptional case:

$$\tilde{t} = 0: \quad \alpha_1 = L, \qquad \alpha_2 = \lambda_P,$$

$$\tilde{t} = 1: \quad \alpha_1 = \tau_Q, \qquad \alpha_2 = L,$$

with $L = \min(\tau_Q, \lambda_p)$. Proceeding in the manner of O'Malley and Keller we obtain the following.

Theorem 2.2. *Consider the differential equation*

$$\epsilon My + Ny = 0$$

(*with My and Ny as defined above*) *subject to the boundary conditions*

$$y^{(\lambda_i)}(0) = l_i, \qquad i = 1, 2, \ldots, r,$$

$$y^{(\tau_i)}(1) = l_{r+i}, \qquad i = 1, 2, \ldots, m - r,$$

where

$$m > \lambda_1 > \lambda_2 > \cdots > \lambda_r \geqslant 0,$$

and

$$m > \tau_1 > \tau_2 > \cdots > \tau_{m-r} \geqslant 0.$$

Further, let $z(x)$ *satisfy the reduced problem*

$$\begin{aligned} Nz &= 0, \\ z^{(\lambda_i)}(0) &= l_i, \qquad i = P+1, P+2, \ldots, r, \\ z^{(\tau_j)}(1) &= l_{r+j}, \qquad j = Q+1, Q+2, \ldots, m-r, \end{aligned}$$

where $P \leqslant r$ *and* $Q \leqslant m - r$.

Suppose

1. *The cancellation law is meaningful,*
2. *The reduced problem has a unique solution* $z(x)$,
3. (a) $\lambda_1, \lambda_2, \ldots, \lambda_P$ *are distinct modulo* $(m - n)$,
   (b) $\tau_1, \tau_2, \ldots, \tau_Q$ *are distinct modulo* $(m - n)$.

Then, a unique solution $y(x)$ *exists for* ϵ *sufficiently small and has the form*

$$y(x) = \kappa^{\alpha_1} \sum_{k=1}^{0} y_k(x, \kappa) + \kappa^{\alpha_2} \sum_{k=Q+1}^{m-n} y_k(x, \kappa) + u(x, \kappa)$$

where $u(x, \kappa)$ *is a regular perturbation of* $z(x)$.

4. *In the exceptional case, also suppose*
   (a) $\alpha_1 > 0$ *if* $\tilde{t} = 1$,
   (b) $\alpha_2 > 0$ *if* $\tilde{t} = 0$.

Then

$$y(x) \sim z(x) \qquad \text{on} \quad 0 < x < 1.$$

Note. (1) These results extend the conclusions of Wasow [109] (cf., e.g., the exceptional case) and are, in turn, extended in O'Malley and Keller [88]. These papers all deal with scalar equations with separated (or uncoupled) boundary conditions. An example with coupled boundary conditions is analyzed in Friedman [34].

(2) That hypotheses 1, 2, 3, and 4 are essentially necessary can be seen by considering a series of problems for constant coefficient equations whose solutions are, except for very special boundary values (e.g., all zero)

divergent as $\epsilon \to 0$. See, e.g., Wasow [109] and O'Malley [81] for lists of examples.

(3) This procedure of finding an asymptotic fundamental system and taking a suitable linear combination of these asymptotic solutions to obtain the limiting solution (if it exists) is the method used by Wasow [109] to obtain the first term in each such product series of the linear combination. This method, without modification (as Latta [63]) points out), is extremely laborious for obtaining further terms in the expansions needed. Furthermore, one does not always need a complete system of asymptotic fundamental solutions to obtain the uniform expansion needed. For example, for $O(\kappa^L)$ accuracy, only the first L terms of $u(x, \kappa)$ are of interest (and the solution y—to this order of approximation—is a regular perturbation of $z(x)$ uniformly on $0 \leqslant x \leqslant 1$). Likewise, if $L = \lambda_p$ and $\tilde{t} = 1$, for $O(\kappa^{\tau Q})$ accuracy, no terms of the expansions for $y_1, y_2, \ldots, y_Q$ need be calculated.

A Modified Approach

In dealing with some more complicated singular perturbation problems featuring turning-points and nonlinearities, Cochran [14] introduced new variables (or stretching transformations) in terms of which he obtained the asymptotic solutions desired. To illustrate the technique, which sometimes proceeds heuristically, we reconsider the linear problem

$$\begin{cases} \epsilon y'' + a(x)\, y' + b(x)\, y = 0, \\ y(0), \quad y(1) \qquad \text{prescribed}, \\ \text{for} \quad a(x) > 0 \qquad \text{on} \quad 0 \leqslant x \leqslant 1. \end{cases} \tag{2.3}$$

Experience (and our previous calculation) would predict that the solution of this boundary value problem will feature nonuniform convergence in the vicinity of $x = 0$ and that this boundary layer behavior may be described in terms of the two variables x and $1/\epsilon \int_0^x a(s)\, ds$.

Thus, we introduce the stretching transformation

$$\eta = \frac{1}{\epsilon} \int_0^x [g_0(s) + \epsilon g_1(s)]\, ds \tag{2.11}$$

with the g_i's undetermined, but with g_0 positive, and we formally set

$$y(x) \equiv y(x, \eta) = \sum_{k=0}^{\infty} y_k(x, \eta)\, \epsilon^k, \tag{2.12}$$

where we ask that the functions y_k be bounded independent of ϵ for all values of x and η and that certain "secular" terms be eliminated. We proceed to demonstrate the procedure.

Formally, substitute (2.12) into (2.3) and equate to zero the coefficient of each power of ϵ. The coefficient of ϵ^{-1}, the lowest power appearing, yields

$$g_0^2(y_{0\eta\eta} + (a/g_0)\, y_{0\eta}) = 0. \tag{2.13a}$$

Integrating twice with respect to η, we take

$$y_0(x, \eta) = A_0(x) + B_0(x)\, e^{-(a/g_0)\eta}.$$

Since g_1 is arbitrary, however, we take $B_0(x) = b_0$, a constant, without loss of generality. The coefficient of ϵ^0 yields

$$\begin{aligned} y_{1\eta\eta}g_0^2 + y_{1\eta}ag_0 + 2y_{0x\eta}g_0 + 2y_{0\eta\eta}g_1g_0 \\ + y_{0\eta}g_0' + ay_{0x} + ag_1y_{0\eta} + by_0 = 0. \end{aligned} \tag{2.13b}$$

Using (2.13a) and integrating with respect to η,

$$\begin{aligned} g_0^2[y_1e^{(a/g_0)\eta}]_\eta + B_0\left[2g_0\left(-\frac{a}{g_0}\right)'\eta - g_1a + g_0'\right] + \eta e^{(a/g_0)\eta}[aA_0' + bA_0] \\ + e^{(a/g_0)\eta}\left[\int_0^\eta \left[aB_0\left(-\frac{a}{g_0}\right)'\eta e^{-(a/g_0)\eta} + bB_0e^{-(a/g_0)\eta}\right] d\eta\right] = \tilde{A}_1(x)\, e^{(a/g_0)\eta}. \end{aligned}$$

For boundedness of y_1, we need to pick

$$aA_0' + bA_0 = 0,$$

so

$$A_0(x) = a_0 \exp\left[-\int_1^x \frac{b(s)}{a(s)}\, ds\right].$$

Likewise, to eliminate secular terms of the form $\eta^2 \exp[-(a/g_0)\eta]$ in y_1, we require that

$$g_0B_0(a/g_0)' = 0$$

and, therefore, that $g_0(x) = a(x)$. Thus,

$$a^2y_1e^\eta + b_0\eta[-g_1a + a' - b] = \tilde{A}_1(x)\, e^\eta + \tilde{B}_1(x).$$

To eliminate a further secular term of the form $\eta e^{-\eta}$, we set

$$g_1(x) = \frac{a'(x)}{a(x)} - \frac{b(x)}{a(x)}$$

and obtain

$$y_1(x, \eta) = A_1(x) + B_1(x)\, e^{-\eta}.$$

Thus

$$\eta = \frac{1}{\epsilon} \int_0^x \left[a(s) + \epsilon \left((\ln a(s))' - \frac{b(s)}{a(s)}\right)\right] ds$$

and

$$y_0(x, \eta) = a_0 \exp\left[-\int_1^x \frac{b(s)}{a(s)}\, ds\right] + \frac{b_0}{a(x)} \exp\left[-\frac{1}{\epsilon}\int_0^x \left(a(s) - \frac{b(s)}{a(s)}\right) ds\right],$$

where we need to select $a_0 = y(1)$ and

$$b_0 = a(0)\left[y(0) - y(1) \exp\left[-\int_1^0 \frac{b(s)}{a(s)}\, ds\right]\right]$$

in order to satisfy the boundary conditions. Appropriate specialization of the form of $y_1(x, \eta)$ has thus yielded a complete determination of $y_0(x, \eta)$. Higher-order terms are successively obtained in a similar manner.

Surely, this example was more easily solved previously. However, being familiar with this procedure we can solve more complicated examples. Specifically, Cochran [14] successfully attacked the nonlinear problem

$$\epsilon y'' + yy' - y = 0,$$
$$y(0), \quad y(1) \qquad \text{prescribed}$$

and the turning-point problem

$$\epsilon y'' - (\tfrac{1}{2} - x)\, y' - y = 0,$$
$$y(0), \quad y(1) \qquad \text{prescribed},$$

among others, and O'Malley [85] considered nonlinear problems of the form

$$\epsilon y'' + f(x, y)\, y' + g(x, y) = 0,$$
$$y(0), \quad y(1) \qquad \text{prescribed},$$

In particular, complete expansions valid for $0 \leqslant x \leqslant 1$ were obtained in the special case where f and g are infinitely differentiable and where $f(x, y) = a(x) \neq 0$. Note that similar analysis is reported in Erdélyi [29] and an engineering application is given in O'Malley [86]. Note also that earlier results on this problem were obtained by Coddington and Levinson [16], Wasow [113], Erdélyi [26], and Willett [118].

3. Singular Perturbation of Eigenvalue Problems

The Vibrating String

Recall the eigenvalue problem which arises in considering the vibrations of a string with clamped end points (refer, e.g., to Rayleigh [93]). If the string has negligible stiffness, we consider the problem

$$\begin{cases} -y'' = \lambda^2 y, \\ y(0) = y(1) = 0 \end{cases} \tag{3.1}$$

on the x-interval $[0, 1]$. If, however, stiffness effects are introduced, we consider the higher-order problem

$$\begin{cases} \epsilon y^{\text{IV}} - y'' = \lambda^2 y, \\ y(0) = y(1) = y'(0) = y'(1) = 0, \end{cases} \tag{3.2}$$

with ϵ positive and proportional to the stiffness. As the string stiffness goes to zero ($\epsilon \to 0$), we expect that the eigenvalues and eigenfunctions of (3.2) should converge (but not necessarily uniformly) to those of (3.1), i.e., $n^2\pi^2$ and $\sin n\pi x$, respectively, for $n = 1, 2, \ldots$.

We associate with (3.2) the auxiliary equation

$$\epsilon\rho^4 - \rho^2 - \lambda^2 = 0$$

which has two singular roots (as $\epsilon \to 0$) of the form $\pm 1/\kappa + O(\kappa)$ for

$$\kappa = \epsilon^{1/2} > 0$$

and λ^2 bounded. This, naturally, leads us to introduce stretching coordinates x/κ and $(1 - x)/\kappa$ and to formally let

$$y(x, \epsilon) = A(x, \kappa) + B(x, \kappa)\, e^{-x/\kappa} + C(x, \kappa)\, e^{-(1-x)/\kappa} \tag{3.3}$$

and

$$\lambda^2 \sim \lambda^2(\kappa) \equiv \sum_{j=0}^{\infty} \lambda_j^{\,2} \kappa^j$$

where A, B, and C have power series expansions in κ with jth coefficients $a_j(x)$, $b_j(x)$, and $c_j(x)$, respectively.

Substituting (3.3) into (3.2) and equating coefficients of the functions 1, $e^{-x/\epsilon}$ and $e^{-(1-x)/\epsilon}$, we ask that

$$A'' + \lambda^2 A = \kappa^2 A^{\mathrm{IV}},$$

$$2B' = \kappa(5B'' - \lambda^2 B) - 4\kappa^2 B''' + \kappa^3 B^{\mathrm{IV}},$$

and

$$2C' = -\kappa(5C'' - \lambda^2 C) - 4\kappa^2 C''' - \kappa^3 C^{\mathrm{IV}}.$$

This, in turn, implies the following differential equations for the coefficients $a_j(x)$, $b_j(x)$, and $c_j(x)$:

$$a_j'' + \lambda_0^{\,2} a_j = a_{j-2}^{\mathrm{IV}} - \sum_{l=1}^{j} \lambda_l^{\,2} a_{j-l} \equiv -\lambda_j^{\,2} a_0 + \alpha_{j-1},$$

$$2b_j' = 5b_{j-1}'' - \sum_{l=0}^{j-1} \lambda_l^{\,2} b_{j-1-l} - 4b_{j-2}''' + b_{j-3}^{\mathrm{IV}} \equiv 2\beta_{j-1},$$

and

$$2c_j' = -5c_{j-1}'' + \sum_{l=0}^{j-1} \lambda_l^{\,2} c_{j-1-l} - 4c_{j-2}''' - c_{j-3}^{\mathrm{IV}} \equiv 2\gamma_{j-1},$$

for all integers $j \geqslant 0$ where a_j, b_j, and c_j with negative coefficients are defined to be identically zero.

Proceeding further, formally

$$0 = y(0, \epsilon) \sim A(0, \kappa) + B(0, \kappa),$$

$$0 = \kappa y'(0, \epsilon) \sim \kappa[A'(0, \kappa) + B'(0, \kappa)] - B(0, \kappa),$$

$$0 = y(1, \epsilon) \sim A(1, \kappa) + C(1, \kappa),$$

and

$$0 = \kappa y'(1, \epsilon) \sim \kappa[A'(1, \kappa) + C'(1, \kappa)] + C(1, \kappa),$$

which leads to the successively determined boundary conditions:

$$a_j(0) = -b_j(0) = -a_{j-1}'(0) - b_{j-1}'(0),$$

and

$$a_j(1) = -c_j(1) = a'_{j-1}(1) + c'_{j-1}(1).$$

From these formal procedures, we can inductively determine the expansions $A(x, \kappa)$, $B(x, \kappa)$, $C(x, \kappa)$ and $\lambda^2(\kappa)$. For $j = 0$,

$$b'_0(x) = 0, \qquad b_0(0) = 0,$$
$$c'_0(x) = 0, \qquad c_0(1) = 0$$

imply that $b_0(x) = c_0(x) \equiv 0$, while

$$\begin{aligned} &a''_0 + \lambda_0{}^2 a_0 = 0, \\ &a_0(0) = a_0(1) = 0 \end{aligned} \tag{3.4a}$$

implies that there is a nontrivial solution for $a_0(x)$ only if $\lambda_0 = n\pi$, $n = 1, 2, 3, \ldots$, and then $a_0(x) = A_0 \sin n\pi x$, A_0 a constant. For a unique, nontrivial determination of $a_0(x)$ we normalize by asking that

$$\int_0^1 a_0{}^2(x)\, dx = 1$$

which fixes $A_0 = \sqrt{2}$.

For $j = 1$:

$$2b'_1 = 5b''_0 - \lambda_0{}^2 b_0\,, \qquad b_1(0) = a'_0(0) + b'_0(0),$$
$$2c'_1 = -5c''_0 + \lambda_0{}^2 c_0\,, \qquad c_1(1) = -a'_0(1) - c'_0(1)$$

imply that $b_1(x) = \sqrt{2}\, n\pi$ and $c_1(x) = (-1)^n \sqrt{2}\, n\pi$, while

$$\begin{cases} a''_1 + \lambda_0{}^2 a_1 = -\lambda_1{}^2 a_0\,, \\ a_1(0) = -b(0), \qquad a_1(1) = -c_1(1), \end{cases} \tag{3.4b}$$

and (3.4a) imply that

$$[a'_0(x)\, a_1(x) - a'_1(x)\, a_0(x)]' = \lambda_1{}^2 a_0{}^2(x).$$

Integrating from 0 to 1, then,

$$\lambda_1{}^2 = 4n^2\pi^2.$$

Integrating (3.4b) yields

$$a_1(x) = A_1 \sin n\pi x + (B_1 + 2\sqrt{2}\, n\pi x) \cos n\pi x$$

and applying either boundary condition fixes $B_1 = -\sqrt{2}\, n\pi$. To obtain $a_1(x)$ uniquely, we ask that it be orthogonal to $a_0(x)$, i.e.,

$$\int_0^1 a_1(x)\, a_0(x)\, dx = 0,$$

which yields $A_1 = \sqrt{2}$. Hence, at the second step, we have

$$\lambda^2(\kappa) = n^2\pi^2 + 4\kappa n^2\pi^2 + \cdots,$$

$$A(x, \kappa) = \sqrt{2} \sin n\pi x + \sqrt{2}\, \kappa[\sin n\pi x - n\pi \cos n\pi x + 2n\pi x \cos n\pi x] + \cdots,$$

$$B(x, \kappa) = 0 + \sqrt{2}\, \kappa n\pi + \cdots,$$

and

$$C(x, \kappa) = 0 + (-1)^n \sqrt{2}\, \kappa n\pi + \cdots.$$

At jth step: If $a_l(x)$, $b_l(x)$, $c_l(x)$ and λ_l^2 are known for $l < j$, the differential equations and boundary conditions imply that

$$b_j(x) = b_j(0) + \int_0^x \beta_{j-1}(x)\, dx,$$

$$c_j(x) = c_j(1) + \int_1^x \gamma_{j-1}(x)\, dx,$$

and

$$a_j'' + \lambda_0^2 a_j = -\lambda_j^2 a_0 + \alpha_{j-1}(x). \tag{3.4c}$$

From (3.4a) and (3.4c)

$$[a_0'(x)\, a_j(x) - a_j'(x)\, a_0(x)]' = \lambda_j^2 a_0^2(x) - a_0(x)\, \alpha_{j-1}(x)$$

so that

$$\lambda_j^2 = a_0'(1)\, a_j(1) - a_0'(0)\, a_j(0) + \int_0^1 a_0(x)\, \alpha_{j-1}(x)\, dx.$$

Moreover, integrating (3.4c) and applying the boundary conditions, $a_j(x)$ is determined up to an arbitrary term $A_j \sin nx$. To fix A_j uniquely, we merely require

$$\int_0^1 a_j(x)\, a_0(x)\, dx = 0.$$

Thus, all coefficients in the series $A(x, \kappa)$, $B(x, \kappa)$, $C(x, \kappa)$ and $\lambda^2(\kappa)$ may be successively obtained. That the resulting expansions for $y(x, \epsilon)$

and λ^2 are, indeed, asymptotically correct is proved in Miranker [74]. Although the expansion $\lambda^2(\kappa)$ is analytic at $\kappa = 0$, the eigenvalues are actually of the form

$$\lambda^2 = n^2\pi^2 + S_n(\kappa, e^{-1/\kappa}), \qquad n = 1, 2,...$$

where $S_n(\xi, \eta)$ is a power series in ξ and η with $S_n(0, 0) = 0$. Since $e^{-1/\kappa} \sim 0$, λ^2 is asymptotically represented simply by a power series in κ.

Higher-Order Problems

Generalizing, we consider eigenvalue problems for the equation

$$\epsilon My + Ny = \lambda y \tag{3.5}$$

where M and N are differential operators on the x-interval $[0, 1]$ of even orders $2m$ and $2n$, respectively, with $m > n$. Such problems have been dealt with in Moser [77]. Specifically, let

$$My = \sum_{j=1}^{2m} \alpha_j(x)\, y^{(j)}(x)$$

and

$$Ny = \sum_{j=1}^{2n} \beta_j(x)\, y^{(j)}(x),$$

where

$$(-1)^m\, \alpha_{2m}(x) > 0 \qquad \text{and} \qquad (-1)^n\, \beta_{2n}(x) > 0 \tag{3.6}$$

throughout $[0, 1]$. Further, let m boundary conditions be prescribed at each end point of the form

$$\begin{cases} S_i y = y^{(\sigma_i)}(0) + \sum\limits_{j<\sigma_i} s_{ij} y^{(j)}(0) = 0, \\ T_i y = y^{(\tau_i)}(1) + \sum\limits_{j<\tau_i} t_{ij} y^{(j)}(1) = 0, \qquad i = 1, 2,..., m, \end{cases} \tag{3.7}$$

where

$$2m > \sigma_1 > \sigma_2 > \cdots > \sigma_m \geqslant 0$$

and

$$2m > \tau_1 > \tau_2 > \cdots > \tau_m \geqslant 0.$$

We associate with (3.5) the auxiliary equation

$$\epsilon \sum_{j=1}^{2m} \alpha_j(x) D^j + \sum_{j=1}^{2n} \beta_j(x) D^j = \lambda, \tag{3.8}$$

which has $2(m - n)$ singular roots (as $\epsilon \to 0$) of the form

$$\frac{d(x, \kappa)}{\kappa} = \frac{1}{\kappa}[g(x) + O(\kappa^{2m})],$$

where

$$\kappa = \epsilon^{1/2(m-n)} > 0$$

and

$$g(x) = \left(-\frac{\beta_{2n}(x)}{\alpha_{2m}(x)}\right)^{1/2(m-n)}.$$

Let $g_1, g_2, \ldots, g_{m-n}$ enumerate those roots g with negative real parts, and let $g_{m-n+1}, g_{m-n+2}, \ldots, g_{2(m-n)}$ enumerate the roots with positive real parts. [Hypothesis (3.6) implies that no determination of g has real part zero.] Since, then, $m - n$ singular roots of the characteristic equation approach $-\infty$ as $\epsilon \to 0$ while $m - n$ other roots approach $+\infty$, we are naturally led to define the "reduced eigenvalue problem," i.e., the eigenvalue problem for $\epsilon = 0$, by

$$\begin{cases} Ny = \lambda y, \\ S_i y = 0, \qquad i = m - n + 1, \ldots, m, \\ T_i y = 0 \end{cases} \tag{3.9}$$

where $m - n$ boundary conditions (involving the highest-order derivatives) have been omitted at each end point. Clearly, for (3.9) to be reasonable, we must require that

$$\sigma_{m-n+1} < 2n \qquad \text{and} \qquad \tau_{m-n+1} < 2n. \tag{3.10}$$

To determine the eigenvalues of the reduced problem (3.9), we introduce a fundamental system $z_1, z_2, \ldots, z_{2n}$ of the reduced equation $Nz = \lambda z$, noting that each z_i is an entire function of λ. The eigenvalues λ_0 of the reduced problem (3.9) are, then, the zeros of the entire function

$$F(\lambda) = \det \begin{bmatrix} S_i z_k \\ T_i z_k \end{bmatrix} \qquad (i = m - n + 1, \ldots, m, \quad k = 1, 2, \ldots, 2n)$$

with corresponding (normalized) eigenfunctions $u_0(x)$.

A complete set of fundamental asymptotic solutions of the differential equation (3.5) may be formally constructed of the form

$$A_j(x, \kappa), \qquad j = 1, 2, \ldots, 2n,$$

$$B_j(x, \kappa) \exp \left[\frac{1}{\kappa} \int_0^x g_j(s)\, ds\right], \qquad j = 1, 2, \ldots, m - n,$$

and

$$C_j(x, \kappa) \exp \left[\frac{1}{\kappa} \int_1^x g_{m-n+j}(s)\, ds\right], \qquad j = 1, 2, \ldots, m - n$$

where A_j, B_j, and C_j represent power series in κ with variable coefficients $a_{jl}(x)$, $b_{jl}(x)$, and $c_{jl}(x)$, respectively. Without loss of generality, the series $A_j(x, \kappa)$ may be considered as real with $a_{j0}(x) = z_j$; and for λ real and $\bar{g}_p(x) = g_q(x)$, $\overline{B_p} = B_q$ or $\overline{C_{p-m+n}} = C_{q-m+n}$. Continuing, we obtain the following.

Theorem 3.1. *If λ_0 is a simple eigenvalue of the reduced eigenvalue problem (3.9) (i.e., if $F(\lambda_0) = 0, F'(\lambda_0) \neq 0$) and if*

$$\sigma_1, \sigma_2, \ldots, \sigma_{m-n}$$

are distinct modulo $2(m - n)$ and the same is true for

$$\tau_1, \tau_2, \ldots, \tau_{m-n},$$

then, for sufficiently small $\epsilon > 0$, there exists a uniquely determined eigenvalue $\lambda = \lambda(\epsilon)$ in a neighborhood of $\lambda = \lambda_0$ for the original eigenvalue problem (3.5), (3.7). Further, this eigenvalue has an asymptotic expansion of the form

$$\lambda \sim \sum_{j=0}^{\infty} \Lambda_j \kappa^j$$

where

$$\kappa = \epsilon^{1/2(m-n)} \qquad \textit{and} \qquad \Lambda_0 = \lambda_0 .$$

The corresponding eigenfunction y belonging to $\lambda = \lambda(\epsilon)$ is asymptotically of the form

$$y \sim A(x, \kappa) + \kappa^{\sigma_{m-n}} \sum_{j=1}^{m-n} B_j(x, \kappa) \exp \left[\frac{1}{\kappa} \int_0^x g_j(s)\, ds\right]$$

$$+ \kappa^{\tau_{m-n}} \sum_{j=1}^{m-n} C_j(x, \kappa) \exp \left[\frac{1}{\kappa} \int_1^x g_{m-n+j}(s)\, ds\right]$$

throughout the closed interval [0, 1]. *Here,* $A(x, \kappa)$ *is determined uniquely up to a factor depending on* κ. *Making an appropriate choice of this factor,*

$$A(x, \kappa) \equiv \sum_{j=0}^{\infty} a_j(x)\, \kappa^j$$

with $a_0(x) = u_0(x)$, *the normalized eigenfunction corresponding to* λ_0.

Moser [77] considers the case where the operators M and N are formally self-adjoint and the boundary conditions (3.7) are self-adjoint with respect to both M and N. He first shows that these requirements are sufficient to guarantee that (3.10) holds. General results for self-adjoint operators (see Rellich [94]), then, imply the existence of infinitely many discrete eigenvalues $\lambda_n(\epsilon)$, $n = 1, 2, \ldots$, which are analytic functions of ϵ for $\epsilon > 0$. Then, by investigating the behavior of $\lambda_n(\epsilon)$ in the neighborhood of $\epsilon = 0$, Moser was able to show the following.

Theorem 3.2. *Under the general assumptions of Theorem* 3.1 *and in the self-adjoint case, where the reduced eigenvalue problem* (3.9) *has only simple eigenvalues,*

$$\lim_{\epsilon \to 0} \lambda_n(\epsilon) = \lambda_n(0)$$

exists for each $n = 1, 2, \ldots$. *Moreover, the values* $\{\lambda_n(0)\}$ *form the full set of eigenvalues of* (3.9). *Thus, the eigenvalues* $\lambda_n(\epsilon)$ *have asymptotic expansions for small values of* ϵ *given by Theorem* 3.1.

Handelman, Keller, and O'Malley [42] study singular perturbations of eigenvalue problems for linear differential equations where the boundary conditions involve the small parameter ϵ and the eigenvalue parameter λ in such a way that the highest order of differentiation involved may be smaller at $\epsilon = 0$ than for $\epsilon > 0$. This reduction in order of the boundary conditions has a pronounced effect on the asymptotic behavior of the eigenvalues and eigenfunctions. In particular, it is shown that eigenvalues of the full problem can, under appropriate conditions, have limits as ϵ approaches zero which are not eigenvalues of the reduced problem. Among other papers concerned with eigenvalue problems, the reader should note Harris [44], which analyzes systems of first-order equations, and Boyce and Handelman [8] and Handelman and Keller [41], which discuss applications.

4. Two-Parameter Singular Perturbation Problems for Ordinary Differential Equations

Thus far we have considered the limiting behavior (as $\epsilon \to 0$) of solutions y_ϵ of boundary value problems for equations of the form

$$\epsilon My + Ny = 0,$$

where M and N are ordinary differential operators with m, the order of M, greater than n, that of N. Physical problems are frequently of a more complicated nature, however, and often involve several small interrelated parameters, e.g., ϵ, ϵ^3, $\epsilon \log^2 \epsilon$, $\epsilon^{1/2}$, or even $f(\epsilon)$. Thus, we are naturally led to consider two-parameter singular perturbation problems for equations of the form

$$\epsilon My + \mu Ny + Ly = 0,$$

where ϵ and μ are small, positive, interrelated parameters simultaneously approaching zero and $m > n > l$ where m, n, and l are the orders of the differential operators M, N, and L, respectively. Such problems, as well as analogous problems for partial differential equations, are considered in O'Malley [81] and the subsequent papers O'Malley [82], which considers second order problems, and O'Malley [83], which considers higher-order problems.

Second-Order Problems

A. *An Initial-Value Problem*

Consider, first, the constant coefficient initial-value problem.

$$\epsilon y'' + \mu a y' + by = 0,$$

$$y(0), \quad y'(0) \qquad \text{prescribed}$$

on the x-interval $[0, 1]$ for a and b positive constants.

Case 1: $\epsilon/\mu^2 \to 0$ *as* $\mu \to 0$.

The general solution is

$$y(x) = \frac{1}{2\sqrt{1-\dfrac{4\epsilon b}{\mu^2 a^2}}}\left\{\left[y(0)\left(1+\sqrt{1-\frac{4\epsilon b}{\mu^2 a^2}}+\frac{2\epsilon y'(0)}{\mu a}\right)\right]\right.$$

$$\cdot \exp\left[-\frac{\mu a}{2\epsilon}\left(1-\sqrt{1-\frac{4\epsilon b}{\mu^2 a^2}}\right)x\right]$$

$$+\left[y(0)\left(-1+\sqrt{1-\frac{4\epsilon b}{\mu^2 a^2}}\right)-\frac{2\epsilon y'(0)}{\mu a}\right]$$

$$\left.\cdot \exp\left[-\frac{\mu a}{2\epsilon}\left(1+\sqrt{1-\frac{4\epsilon b}{\mu^2 a^2}}\right)x\right]\right\},$$

which converges to $u(x) \equiv 0$ on every interval $[\delta, 1]$, $\delta > 0$, but convergence is nonuniform at $x = 0$ for $y(0) \neq 0$. Further, the limiting behavior of $y(x)$ is determined primarily by the factors in the exponents, which are, simply, the roots of the auxiliary polynomial

$$\epsilon D^2 + \mu a D + b = 0.$$

Note, that these roots both approach $-\infty$ as $\mu \to 0$, that they can be expanded in powers of the small parameter ϵ/μ^2, and that the exponential decay of $y(x)$ is determined by the singular portion of these roots, e.g., if $\epsilon = \mu^4$, by $-b/\mu a$ and $-1/\mu^3(a - \mu^2 b/a)$.

Case 2: $\mu^2/\epsilon \to 0$ *as* $\epsilon \to 0$.

The solution for $\epsilon > 0$ is

$$y(x) = e^{-\frac{\mu a x}{2\epsilon}}\left\{\left[\frac{y'(0)\sqrt{\epsilon}+\dfrac{\mu a}{2\sqrt{\epsilon}}y(0)}{\sqrt{b-\dfrac{\mu^2 a^2}{4\epsilon}}}\right]\sin\left(\frac{x}{\sqrt{\epsilon}}\sqrt{b-\frac{\mu^2 a^2}{4\epsilon}}\right)\right.$$

$$\left.+\,y(0)\cos\left(\frac{x}{\sqrt{\epsilon}}\sqrt{b-\frac{\mu^2 a^2}{4\epsilon}}\right)\right\}$$

and is therefore oscillatory as $\epsilon \to 0$ unless $\epsilon/\mu = o(1)$. Then, however, y converges nonuniformly to zero in (0, 1] as $\epsilon \to 0$, as might be expected since the auxiliary polynomial has conjugate complex solutions

$$-\frac{\mu a}{2\epsilon} \pm \frac{i}{\sqrt{\epsilon}} \sqrt{b - \frac{\mu^2 a^2}{4\epsilon}}\,.$$

whose real part, in this case, approaches $-\infty$ as $\epsilon \to 0$.

With this background, we can predict the limiting behavior for the solution of the variable coefficient initial-value problem

$$\begin{cases} \epsilon y'' + \mu a(x)\, y' + b(x)\, y = 0, \\ y(0), \quad y'(0) \qquad \text{prescribed} \\ \text{for} \quad a(x) > 0, \quad b(x) > 0 \qquad \text{and} \qquad x \in [0, 1] \end{cases} \tag{4.1}$$

by considering the roots of the auxiliary equation

$$\epsilon D^2 + \mu a(x)\, D + b(x) = 0. \tag{4.2}$$

In particular, regular degeneration [i.e., nonuniform convergence on (0, 1) to $u(x) \equiv 0$, the solution of the reduced boundary value problem, as $\epsilon, \mu \to 0$] occurs in both Case 1: $\epsilon/\mu^2 \to 0$ as $\mu \to 0$ and Case 2: μ^2/ϵ and ϵ/μ both $\to 0$ as $\epsilon \to 0$.

For Case 1: Let

$$-\frac{d_1\left(x, \frac{\epsilon}{\mu^2}\right)}{\mu} = -\frac{\mu a(x)}{2\epsilon}\left(1 - \sqrt{1 - \frac{4\epsilon b(x)}{\mu^2 a^2(x)}}\right)$$
$$= -\frac{1}{\mu}\left(\frac{b(x)}{a(x)} + \frac{\epsilon}{\mu^2}\frac{b^2(x)}{a^3(x)} + \frac{2\epsilon^2}{\mu^4}\frac{b^3(x)}{a^5(x)} + \cdots\right)$$

and

$$-\frac{\mu d_2\left(x, \frac{\epsilon}{\mu^2}\right)}{\epsilon} = -\frac{\mu a(x)}{2\epsilon}\left(1 + \sqrt{1 - \frac{4\epsilon b(x)}{\mu^2 a^2(x)}}\right)$$
$$= -\frac{\mu}{\epsilon}\left[a(x) - \frac{\epsilon}{\mu^2}\frac{b(x)}{a(x)} - \frac{\epsilon^2}{\mu^4}\frac{b^2(x)}{a^3(x)} - \cdots\right]$$

be the solutions of the auxiliary equation (4.2), where we note that d_1

and d_2 are both positive for ϵ/μ^2 sufficiently small. Generalizing from the constant coefficient problem, we set

$$y(x) = A\left(x, \frac{\epsilon}{\mu}, \frac{\epsilon}{\mu^2}\right) \exp\left[-\frac{1}{\mu}\int_0^x d_1\left(s, \frac{\epsilon}{\mu^2}\right) ds\right]$$

$$+ B\left(x, \frac{\epsilon}{\mu}, \frac{\epsilon}{\mu^2}\right) \exp\left[-\frac{\mu}{\epsilon}\int_0^x d_2\left(s, \frac{\epsilon}{\mu^2}\right) ds\right].$$

Substituting into (4.1) and formally equating to zero the coefficients of the linearly independent exponentials, we have:

$$\begin{cases} \dfrac{\epsilon}{\mu} A'' - \dfrac{2\epsilon}{\mu^2} d_1 A' - \dfrac{\epsilon}{\mu^2} d_1' A + aA' = 0, \\ \dfrac{\epsilon}{\mu} B'' - 2d_2 B' - d_2' B + aB' = 0. \end{cases} \tag{4.3}$$

Thus, we take A and B to have double power-series expansions in ϵ/μ and ϵ/μ^2 with variable coefficients $a_{rs}(x)$ and $b_{rs}(x)$, respectively. Moreover, these coefficients can be successively obtained uniquely by formally equating coefficients of each pair of powers $\{(\epsilon/\mu)^r, (\epsilon/\mu^2)^s\}$ in (4.3) and formally applying the initial conditions. Thus, we obtain

$$A\left(x, \frac{\epsilon}{\mu}, \frac{\epsilon}{\mu^2}\right) = y(0) + \frac{\epsilon}{\mu^2}\left(y(0)\int_0^x \frac{1}{a(s)}\left(\frac{b(s)}{a(s)}\right)' ds + \frac{y(0)\, b(0)}{a^2(0)}\right) + \frac{\epsilon}{\mu}\frac{y'(0)}{a(0)} + \cdots$$

and

$$B\left(x, \frac{\epsilon}{\mu}, \frac{\epsilon}{\mu^2}\right) = \frac{1}{a(x)}\left[0 - \frac{\epsilon}{\mu^2}\frac{y(0)\, b(0)}{a(0)} - \frac{\epsilon}{\mu} y'(0) + \cdots\right].$$

Note. (1) The functions

$$\frac{1}{\mu}\int_0^x d_1\left(s, \frac{\epsilon}{\mu^2}\right) ds \qquad \text{and} \qquad \frac{\mu}{\epsilon}\int_0^x d_2\left(s, \frac{\epsilon}{\mu^2}\right) ds$$

may be interpreted as boundary layer stretching transformations used to investigate the region of nonuniform convergence near $x = 0$. Roughly speaking, since $A(x, 0, 0) = y(0)$ and $B(x, 0, 0) = 0$, the lowest-order boundary condition dropped by the limiting solution at $x = 0$ [here, $y(0)$], is associated with the less singular stretching transformation

$$\frac{1}{\mu}\int_0^x d_1\left(s, \frac{\epsilon}{\mu^2}\right) ds.$$

(2) The complicated double series may collapse considerably when ϵ is a known function of μ, and this knowledge may be used to advantage in more efficiently determining the formal expansion.

To justify the formal procedure, we should prove the following theorem.

Theorem 4.1. *Let $y(x)$ solve the initial-value problem*

$$\epsilon y'' + \mu a(x)\, y' + b(x)\, y = 0,$$
$$y(0), \quad y'(0) \qquad \textit{prescribed}$$

for $\epsilon/\mu^2 \to 0$ as $\mu \to 0$ and for $x \in [0, 1]$ where $a(x)$ and $b(x)$ are positive.

Further, let the formal procedure outlined above define $a_{rs}(x)$ and $b_{rs}(x)$ uniquely.

Then, for each $N \geqslant 1$,

$$\begin{aligned} y(x) = {} & \Bigg(\sum_{\substack{r,s\geqslant 0\\ r+s\leqslant N}} a_{rs}(x) \left(\frac{\epsilon}{\mu}\right)^r \left(\frac{\epsilon}{\mu^2}\right)^s\Bigg) \exp\left[-\frac{1}{\mu}\int_0^x d_1\left(s, \frac{\epsilon}{\mu^2}\right) ds\right] \\ & + \Bigg(\sum_{\substack{r,s\geqslant 0\\ r+s\leqslant N}} b_{rs}(x) \left(\frac{\epsilon}{\mu}\right)^r \left(\frac{\epsilon}{\mu^2}\right)^s\Bigg) \exp\left[-\frac{\mu}{\epsilon}\int_0^x d_2\left(s, \frac{\epsilon}{\mu^2}\right) ds\right] \\ & + O\left(\left(\frac{\epsilon}{\mu^2}\right)^{N+1}\right) \qquad \textit{throughout} \quad [0, 1]. \end{aligned}$$

Proof. The proof follows immediately from the definitions of d_1, d_2, a_{rs}, and b_{rs}, and the following estimate.

Lemma. *Let $R(x)$ satisfy the initial-value problem*

$$\epsilon R'' + \mu a(x)\, R' + b(x)\, R = \frac{\epsilon}{\mu} g(x, \mu),$$
$$R(0, \mu) = 0, \qquad R'(0, \mu) = \frac{f(\mu)}{\mu},$$

for $\epsilon/\mu^2 \to 0$ as $\mu \to 0$, f and g bounded, and $x \in [0, 1]$ where $a(x)$ and $b(x)$ are positive.

Then $R(x) = O(\epsilon/\mu^2)$ as $\mu \to 0$ uniformly for $x \in [0, 1]$.

Proof. The argument (see O'Malley [82]) is much like that for

Lemma 1 of Chapter 2. One shows that R satisfies an integral equation of the form

$$R(x, \mu) = \int_0^x R(s, \mu)\, K(x, s)\, ds + R_0(x, \mu)$$

where

$$R_0(x, \mu) = O\left(\frac{\epsilon}{\mu^2}\right) = \int_0^x K(x, s)\, ds.$$

Case 2: *μ^2/ϵ and ϵ/μ both $\to 0$ as $\epsilon \to 0$.*

Considering the roots of the auxiliary equation (4.2), we assume the form of solution

$$\begin{aligned} y(x) = \Big[A\left(x, \tau, \frac{\mu}{\tau}\right) \sin\left(\frac{1}{\tau}\int_0^x p\left(s, \left(\frac{\mu}{\tau}\right)^2\right) ds\right) \\ + B\left(x, \tau, \frac{\mu}{\tau}\right) \cos\left(\frac{1}{\tau}\int_0^x p\left(s, \left(\frac{\mu}{\tau}\right)^2\right) ds\right)\Big] \exp\left[-\frac{\mu}{2\tau^2}\int_0^x a(s)\, ds\right], \end{aligned}$$

where

$$\tau = \sqrt{\epsilon}$$

and

$$\begin{aligned} p\left(x, \left(\frac{\mu}{\tau}\right)^2\right) &= \sqrt{b(x) - \left(\frac{\mu}{\tau}\right)^2 \frac{a^2(x)}{4}} \\ &= \sqrt{b(x)}\left[1 - \frac{1}{8}\left(\frac{\mu}{\tau}\right)^2 \frac{a^2(x)}{b(x)} - \cdots\right] \end{aligned}$$

and A and B have asymptotic double power series expansions in τ and μ/τ whose variable coefficients a_{rs} and b_{rs} are successively uniquely determined by the differential equation and the initial conditions.

To show asymptotic convergence, we prove the following

Lemma 1. *Let $R(x)$ satisfy the initial-value problem*:

$$\tau^2 R'' + \mu a(x)\, R' + b(x)\, R = \tau g(x),$$

$$R(0) = 0,$$

$$R'(0) = \frac{f\left(\tau, \frac{\mu}{\tau}\right)}{\tau},$$

where $a(x)$ and $b(x)$ are positive, g and f are bounded, and both τ^2/μ and $\mu/\tau \to 0$ as $\tau \to 0$.

Then R is bounded for all $x \in [0, 1]$.

Proof. (See O'Malley [82].)

Thus, we obtain

Theorem 4.2. *Let $y(x)$ solve the initial-value problem*

$$\epsilon y'' + \mu a(x)\, y' + b(x)\, y = 0,$$

$$y(0), \quad y'(0) \qquad \textit{prescribed},$$

for $x \in [0, 1]$, $a(x)$ and $b(x)$ positive, when ϵ/μ and μ^2/ϵ both $\to 0$ as $\mu \to 0$. Further, let the formal procedure outlined above define $a_{rs}(x)$ and $b_{rs}(x)$. Then, for each $N \geqslant 1$,

$$y(x) = \left\{ \left(\sum_{\substack{r,s\geqslant 0\\ r+s\leqslant N}} a_{rs}(x)\, \tau^r \left(\frac{\mu}{\tau}\right)^s \right) \sin\left(\frac{1}{\tau}\int_0^x p\left(s, \left(\frac{\mu}{\tau}\right)^2\right) ds\right) \right.$$

$$\left. + \left(\sum_{\substack{r,s\geqslant 0\\ r+s\leqslant N}} b_{rs}(x)\, \tau^r \left(\frac{\mu}{\tau}\right)^s \right) \cos\left(\frac{1}{\tau}\int_0^x p\left(s, \left(\frac{\mu}{\tau}\right)^2\right) ds\right) \right\}$$

$$\cdot \exp\left[-\frac{\mu}{2\tau^2}\int_0^x a(s)\, ds\right] + O(\tau^{N+1}) \qquad \textit{uniformly for} \quad x \in [0, 1],$$

where

$$\tau = \sqrt{\epsilon}$$

and

$$p\left(x, \left(\frac{\mu}{\tau}\right)^2\right) = \sqrt{b(x) - \left(\frac{\mu}{\tau}\right)^2 \frac{a^2(x)}{4}}\,.$$

Note. If $a(x) < 0 < b(x)$, the transformation $z = 1 - x$ will allow the reader to solve the terminal-value problem

$$\epsilon y'' + \mu a(x)\, y' + b(x)\, y = 0,$$

$$y(1), \quad y'(1) \qquad \text{prescribed}$$

for $x \in [0, 1]$ for both Case 1 and Case 2.

B. *A Two-Point Boundary Value Problem*

Consider the constant coefficient boundary value problem

$$\epsilon y'' + \mu a y' - by = 0,$$
$$y(0), \quad y(1) \qquad \text{prescribed}$$

for $x \in [0, 1]$ and a and b positive. Note that the sign of the coefficient of y has been reversed from that which occurred in the equation considered above.

For Case 1, where $\epsilon/\mu^2 \to 0$ as $\mu \to 0$, expanding the exact solution, we have

$$y(x) \sim y(1) \exp\left[-\frac{\mu a}{2\epsilon}\left(1 - \sqrt{1 + \frac{4\epsilon b}{\mu^2 a^2}}\right)(x - 1)\right]$$
$$+ y(0) \exp\left[-\frac{\mu a}{2\epsilon}\left(1 + \sqrt{1 + \frac{4\epsilon b}{\mu^2 a^2}}\right) x\right],$$

which converges to $u(x) \equiv 0$ as $\mu \to 0$ on any interval $[\delta, 1 - \delta]$, $\delta > 0$, but converges nonuniformly near $x = 0$ and $x = 1$ unless $y(0) = 0$ and $y(1) = 0$. Further, note that the boundary layer behavior at $x = 0$ is determined by that root

$$-\frac{\mu a}{2\epsilon}\left(1 + \sqrt{1 + \frac{4\epsilon b}{\mu^2 a^2}}\right) = -\frac{\mu}{\epsilon}\left(a + \frac{\epsilon}{\mu^2}\frac{b}{a} - \cdots\right)$$

of the auxiliary equation which approaches $-\infty$ as $\mu \to 0$ while the boundary layer at $x = 1$ is associated with the root

$$-\frac{\mu a}{2\epsilon}\left(1 - \sqrt{1 + \frac{4\epsilon b}{\mu^2 a^2}}\right) = \frac{1}{\mu}\left(\frac{b}{a} - \frac{\epsilon}{\mu^2}\frac{b^2}{a^3} - \cdots\right)$$

which approaches $+\infty$ as $\mu \to 0$. Likewise, for Case 2, where $\mu^2/\epsilon \to 0$ as $\epsilon \to 0$,

$$y(x) \sim y(1) \exp\left[-\sqrt{\frac{b}{\epsilon}}\left(\sqrt{\frac{\mu^2 a^2}{4\epsilon b}} - \sqrt{1 + \frac{\mu^2 a^2}{4\epsilon b}}\right)(x - 1)\right]$$
$$+ y(0) \exp\left[-\sqrt{\frac{b}{\epsilon}}\left(\sqrt{\frac{\mu^2 a^2}{4\epsilon b}} + \sqrt{1 + \frac{\mu^2 a^2}{4\epsilon b}}\right) x\right]$$

and features boundary layer behavior near $x = 0$ and $x = 1$. In particular, if $\mu = o(\epsilon)$,

$$y(x) \sim y(1) \exp\left[\sqrt{\frac{b}{\epsilon}}\,(x-1)\right] + y(0) \exp\left[-\sqrt{\frac{b}{\epsilon}}\,x\right]$$

—the solution of the "semi-reduced" boundary value problem

$$\epsilon z'' - bz = 0,$$
$$z(0) = y(0), \qquad z(1) = y(1).$$

Examine, now, the variable coefficient problem

$$\begin{cases} \epsilon y'' + \mu a(x)\, y' - b(x)\, y = 0, \\ y(0), \quad y(1) \qquad \text{prescribed}, \end{cases} \tag{4.4}$$

for $x \in [0, 1]$ and $a(x)$ and $b(x)$ positive, with auxiliary equation

$$\epsilon D^2 + \mu a(x)\, D - b(x) = 0. \tag{4.5}$$

For Case 1, $\epsilon/\mu^2 \to 0$ as $\mu \to 0$, introduce

$$\frac{d_1\left(x, \frac{\epsilon}{\mu^2}\right)}{\mu} = -\frac{\mu a(x)}{2\epsilon}\left(1 - \sqrt{1 + \frac{4\epsilon b(x)}{\mu^2 a^2(x)}}\right)$$

and

$$-\frac{\mu}{\epsilon}\, d_2\left(x, \frac{\epsilon}{\mu^2}\right) = -\frac{\mu a(x)}{2\epsilon}\left(1 + \sqrt{1 + \frac{4\epsilon b(x)}{\mu^2 a^2(x)}}\right)$$

as the roots of (4.5), with d_1 and d_2 having positive limiting values. Proceeding as usual, we can prove the following.

Theorem 4.3. *Consider the boundary value problem*

$$\epsilon y'' + \mu a(x)\, y' - b(x)\, y = 0,$$
$$y(0), \quad y(1) \qquad \textit{prescribed}$$

for $x \in [0, 1]$ where $a(x)$ and $b(x)$ are positive and for $\epsilon/\mu^2 \to 0$ as $\mu \to 0$.
The boundary value problem has a unique solution $y(x)$. Further,

coefficients $a_{rs}(x)$ and $b_{rs}(x)$ may be successively obtained formally such that, for each $N \geqslant 1$,

$$y(x) = \left(\sum_{\substack{r,s \geqslant 0 \\ r+s \leqslant N}} a_{rs}(x) \left(\frac{\epsilon}{\mu}\right)^r \left(\frac{\epsilon}{\mu^2}\right)^s \right) \exp\left[\frac{1}{\mu} \int_1^x d_1\left(s, \frac{\epsilon}{\mu^2}\right) ds\right]$$

$$+ \left(\sum_{\substack{r,s \geqslant 0 \\ r+s \leqslant N}} b_{rs}(x) \left(\frac{\epsilon}{\mu}\right)^r \left(\frac{\epsilon}{\mu^2}\right)^s \right) \exp\left[-\frac{\mu}{\epsilon} \int_0^x d_2\left(s, \frac{\epsilon}{\mu^2}\right) ds\right]$$

$$+ O\left(\mu \left(\frac{\epsilon}{\mu^2}\right)^{N+1}\right) \qquad \textit{uniformly for} \quad x \in [0, 1].$$

Likewise, for Case 2, we have the following.

Theorem 4.4. *Consider the boundary-value problem*

$$\epsilon y'' + \mu a(x)\, y' - b(x)\, y = 0,$$

$$y(0), \quad y(1) \qquad \textit{prescribed},$$

for $x \in [0, 1]$ where $b(x)$ is positive and for $\mu^2/\epsilon \to 0$ as $\epsilon \to 0$.
The boundary value problem has a unique solution. Further, let

$$-\frac{d_1(x, \mu/\tau)}{\tau} \qquad \textit{and} \qquad \frac{d_2(x, \mu/\tau)}{\tau}$$

be the roots of the auxiliary equation

$$\epsilon D^2 + \mu a(x)\, D - b(x) = 0$$

where $\tau = \sqrt{\epsilon}$ and d_1 and d_2 have positive limiting values. Coefficients $a_{rs}(x)$ and $b_{rs}(x)$ may be successively defined, in the usual manner, such that for each $N \geqslant 1$,

$$y(x) = \left(\sum_{\substack{r,s \geqslant 0 \\ r+s \leqslant N}} a_{rs}(x)\, \tau^r \left(\frac{\mu}{\tau}\right)^s \right) \exp\left[-\frac{1}{\tau} \int_0^x d_1\left(s, \left(\frac{\mu}{\tau}\right)\right) ds\right]$$

$$+ \left[\sum_{\substack{r,s \geqslant 0 \\ r+s \leqslant N}} b_{rs}(x)\, \tau^r \left(\frac{\mu}{\tau}\right)^s \right] \exp\left[\frac{1}{\tau} \int_1^x d_2\left(s, \left(\frac{\mu}{\tau}\right)\right) ds\right]$$

$$+ O(\tau \lambda^{N+1}) \qquad \textit{uniformly for} \quad x \in [0, 1] \quad \textit{where} \quad \lambda = \max(\tau, \mu/\tau).$$

Note.

(1) In proving the previous two theorems, the following maximum-minimum principle is useful:

Lemma. *Suppose z satisfies*

$$\epsilon z'' + \mu a(x)\, z' - b(x)\, z = f(x)$$

for $\epsilon \geqslant 0$ on the x-interval $[0, 1]$ *where $b(x)$ is positive.*
Then

$$|z(x)| \leqslant \operatorname{Max} \left\{ |z(0)|, |z(1)|, \operatorname*{Max}_{x\in[0,1]} \left| \frac{f(x)}{b(x)} \right| \right\}.$$

(2) For Case 2, the coefficient $a(x)$ need not be positive.

(3) The boundary value problem

$$\epsilon y'' - \mu a(x)\, y' - b(x)\, y = 0,$$
$$y(0), \quad y(1) \qquad \text{prescribed},$$

for $x \in [0, 1]$ and $a(x)$ and $b(x)$ positive can likewise be solved in the same manner in the two cases: $\epsilon/\mu^2 \to 0$ as $\mu \to 0$ and $\mu^2/\epsilon \to 0$ as $\epsilon \to 0$.

C. *Other Boundary Value Problems*

The general theory of linear ordinary differential equations implies that for every $\epsilon > 0$, the second-order homogeneous equations considered have two linearly independent solutions and that every solution is a linear combination of these two solutions as well as of every pair of linearly independent solutions, and further, that the solution of the nonhomogeneous equations are obtainable by the method of variation of parameters. This does not imply, however, that any boundary value problem has a unique solution, nor does it imply that the solution of any such boundary value problem has a well-defined limiting behavior as μ and ϵ simultaneously approach zero.

Consider, again, the homogeneous equation

$$\epsilon y'' + \mu a(x)\, y' - b(x)\, y = 0 \tag{4.6}$$

for $x \in [0, 1]$ where $a(x)$ and $b(x)$ are positive and $\epsilon/\mu^2 \to 0$ as $\mu \to 0$. Further, let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of this equation with boundary conditions $y_1(0) = 0, y_1(1) = 1, y_2(0) = 1$, and $y_2(1) = 0$, and note that the asymptotic expansions of y_1 and y_2 follow immediately from Theorem 4.3. Restricting attention to the specific boundary value problem

$$\epsilon y'' + \mu a(x)\, y' - b(x)\, y = 0,$$

$$y'(0), \quad y'(1) \qquad \text{prescribed},$$

we completely determine its asymptotic solution by setting

$$y(x) = C_1(\mu)\, y_1(x) + C_2(\mu)\, y_2(x) \tag{4.7}$$

and obtaining asymptotic expansions for $C_1(\mu)$ and $C_2(\mu)$ by using the known expansions for y_1 and y_2.

Alternately, by *a priori* considerations, one could predict—without calculating—that $C_1(\mu) = O(\mu)$ and is expandable as a double power series in μ and ϵ/μ^2, while $C_2(\mu) = O(\epsilon/\mu)$ and is asymptotically a double series in ϵ/μ and ϵ/μ^2. Thus, knowing that $y(x)$ is given by (4.7) and knowing the form of the expansions for $y_1(x)$ and $y_2(x)$, we may set

$$y(x) \sim \mu A\left(x, \mu, \frac{\epsilon}{\mu^2}\right) \exp\left[\frac{1}{\mu}\int_1^x d_1\left(s, \frac{\epsilon}{\mu^2}\right) ds\right]$$

$$+ \frac{\epsilon}{\mu} B\left(x, \frac{\epsilon}{\mu}, \frac{\epsilon}{\mu^2}\right) \exp\left[-\frac{\mu}{\epsilon}\int_0^x d_2\left(s, \frac{\epsilon}{\mu^2}\right) ds\right].$$

We then proceed to determine A and B in the usual formal manner as double series in the indicated variables for d_1/μ and $-\mu/\epsilon\, d_2$ roots of the auxiliary polynomial (4.5). Clearly, this approach is more efficient than first obtaining the expansions for $y_1(x)$, $y_2(x)$, $C_1(\mu)$, and $C_2(\mu)$ and multiplying the appropriate expansions.

Considering, instead, the initial value problem for Eq. (4.6) where $y(0)$ and $y'(0)$ are prescribed, we find it impossible to construct a solution $y(x)$ which is bounded throughout $[0, 1]$ as $\mu \to 0$. Here, however, the root d_1/μ of the auxiliary polynomial (4.5) approaches $+\infty$ as $\mu \to 0$, so we intuitively expect it to be associated with a boundary layer at $x = 1$, not at $x = 0$.

It is apparent that we need some theorems stating *à priori* which boundary value problems have converging solutions as ϵ and μ simultaneously approach zero and, further, when the limiting behavior is convergent, some *à priori* estimates on the orders of the coefficients $C_i(\mu)$. Restricting attention to the boundary value problems for

$$\epsilon y'' + \mu a(x) y' + b(x) y = 0,$$

on $0 \leqslant x \leqslant 1$ where $a(x) \neq 0 \neq b(x)$ and for $\epsilon/\mu^2 \to 0$ as $\mu \to 0$, we find that the solutions of

(1) the initial-value problem when $a(x)$ and $b(x)$ are positive,
(2) the terminal-value problem when $a(x)$ and $b(x)$ are negative, and
(3) any problem with separated (or unmixed) boundary conditions when $a(x)$ and $b(x)$ have opposite signs

will all converge nonuniformly on $[0, 1]$ (as $\epsilon, \mu \to 0$) to zero (the solution of the reduced equation). Further, complete asymptotic solutions are obtainable. For other boundary conditions, however, the solution of the boundary value problem will, in general, fail to converge (as $\epsilon, \mu \to 0$) as simple, constant coefficient examples easily demonstrate.

D. *Nonhomogeneous Boundary Value Problem*

Consider, now, the nonhomogeneous problem

$$\begin{cases} \epsilon y'' + \mu a(x) y' - b(x) y = R(x), \\ y(0), \quad y(1) \qquad \text{prescribed,} \end{cases} \tag{4.8}$$

for $x \in [0, 1]$ where $a(x)$ and $b(x)$ are positive and $\epsilon/\mu^2 \to 0$ as $\mu \to 0$.

Let $y_1(x)$ and $y_2(x)$ be solutions of the homogeneous equation satisfying the boundary conditions $y_1(0) = 0$, $y_1(1) = 1$, $y_2(0) = 1$, and $y_2(1) = 0$ and let

$$J[y_2, y_1] = \epsilon \exp\left[\frac{\mu}{\epsilon}\int_0^x a(s)\, ds\right] [y_1'(x)\, y_2(x) - y_2'(x)\, y_1(x)].$$

This "conjoint" of y_1 and y_2 is a constant and we use it to represent the solution $y(x)$ of (4.8) by the Green's function formula

$$\begin{aligned} y(x) = &- \frac{y_2(x)}{J[y_2, y_1]} \int_0^x y_1(t) \exp\left[\frac{\mu}{\epsilon}\int_0^t a(s)\, ds\right] R(t)\, dt \\ &- \frac{y_1(x)}{J[y_2, y_1]} \int_x^1 y_2(t) \exp\left[\frac{\mu}{\epsilon}\int_0^t a(s)\, ds\right] R(t)\, dt \\ &+ y(1)\, y_1(x) + y(0)\, y_2(x) \end{aligned}$$

(see, e.g., Friedman [32], Chapter 3).

Formally substituting the expansions for $y_1(x)$ and $y_2(x)$ into this expression (see O'Malley [82]), we obtain

$$y(x) \sim z\left(x, \frac{\epsilon}{\mu}, \frac{\epsilon}{\mu^2}\right) + \tilde{A}\left(x, \frac{\epsilon}{\mu}, \frac{\epsilon}{\mu^2}\right) \exp\left[\frac{1}{\mu} \int_1^x d_1\left(s, \frac{\epsilon}{\mu^2}\right) ds\right]$$
$$+ \tilde{B}\left(x, \frac{\epsilon}{\mu}, \frac{\epsilon}{\mu^2}\right) \exp\left[-\frac{\mu}{\epsilon} \int_0^x d_2\left(s, \frac{\epsilon}{\mu^2}\right) ds\right], \tag{4.10}$$

where $z(x, \epsilon/\mu, \epsilon/\mu^2)$ is an asymptotic solution of the nonhomogeneous equation and a constant multiple of

$$B\left(x, \frac{\epsilon}{\mu}, \frac{\epsilon}{\mu^2}\right) \int_0^x A\left(t, \frac{\epsilon}{\mu}, \frac{\epsilon}{\mu^2}\right) \exp\left[-\frac{\mu}{\epsilon} \int_t^x d_2\left(s, \frac{\epsilon}{\mu^2}\right) ds\right] R(t)\, dt$$
$$+ A\left(x, \frac{\epsilon}{\mu}, \frac{\epsilon}{\mu^2}\right) \int_1^x B\left(t, \frac{\epsilon}{\mu}, \frac{\epsilon}{\mu^2}\right) \exp\left[-\frac{1}{\mu} \int_x^t d_1\left(s, \frac{\epsilon}{\mu^2}\right) ds\right] R(t)\, dt$$

where $\tilde{A}(1, \epsilon/\mu, \epsilon/\mu^2)$ and $\tilde{B}(0, \epsilon/\mu, \epsilon/\mu^2)$ are determined so that the sum asymptotically solves the boundary value problem (4.8). The nonhomogeneous problem has thus been asymptotically solved by the Green's function approach. It should be noted, however, that an asymptotic solution of the nonhomogeneous equation could also be obtained as a regular perturbation (in powers of ϵ and μ) of $-R(x)/b(x)$, the solution of the reduced equation, and that the nonhomogeneous problem could then also be solved by adding an appropriate asymptotic solution of the homogeneous equation.

Higher Order Problems

Now consider boundary value problems for the equation

$$\epsilon My + \mu Ny + Ly = 0 \tag{4.11}$$

on the x-interval $[0, 1]$ where M, N, and L are linear ordinary differential operators having orders m, n, and l, respectively, where $m > n > l \geqslant 0$, and ϵ and μ simultaneously approach zero in an interrelated way. Specifically, let

$$My = y^{(m)} + \alpha_1(x)\, y^{(m-1)} + \cdots + \alpha_k(x)\, y^{(m-k)},$$
$$Ny = \beta(x)[y^{(n)} + \beta_1(x)\, y^{(n-1)} + \cdots + \beta_j(x)\, y^{(n-j)}],$$

and

$$Ly = \gamma(x)[y^{(l)} + \gamma_1(x)\, y^{(l-1)} + \cdots + \gamma_{l-1}(x)\, y' + \gamma_l(x)\, y],$$

where $\beta(x) \neq 0 \neq \gamma(x)$. Further, let us consider (4.11) subject to the boundary conditions

$$\begin{cases} y^{(\lambda_i)}(0) = l_i, & i = 1, 2,..., r, \\ y^{(\tau_i)}(1) = l_{r+i}, & i = 1, 2,..., m - r, \end{cases} \tag{4.12}$$

where

$$m > \lambda_1 > \lambda_2 > \cdots > \lambda_r \geqslant 0,$$

and

$$m > \tau_1 > \tau_2 > \cdots > \tau_{m-r} \geqslant 0.$$

As usual, we associate with the differential equation (4.11) its auxiliary equation

$$\epsilon\left[D^m + \sum_{i=1}^{k} \alpha_i(x)\, D^{m-i}\right] + \mu\beta(x)\left[D^n + \sum_{i=1}^{j} \beta_i(x)\, D^{n-i}\right] + \gamma(x)\left[D^l + \sum_{i=1}^{l} \gamma_i(x)\, D^{l-i}\right] = 0 \tag{4.13}$$

and use its singular (as $\epsilon, \mu \to 0$) roots to obtain $m - l$ asymptotic solutions of (4.11). Further, we call equation (4.11) exceptional if this set of $(m - l)$ singular roots of (4.13) contains members with purely imaginary singular parts. Otherwise, (4.11) is called nonexceptional and $m - l = p + q$, where p is the number of singular roots whose real parts have negative limiting values and q is the number of singular roots whose real parts have positive limiting values.

The limiting behavior of the roots of (4.13) as well as the limiting behavior of the solution of the boundary value problem (4.11), (4.12) are completely different in the two cases

$$(1) \quad \frac{\epsilon}{\mu^{((m-l)/(n-l))}} \to 0 \quad \text{as} \quad \mu \to 0, \qquad \text{and} \qquad (2) \quad \frac{\mu^{((m-l)/(n-l))}}{\epsilon} \to 0$$

as $\epsilon \to 0$. Thus, they are considered separately below. A more complete analysis, including coverage of the problem when (4.11) is exceptional and when $\epsilon = \mu^{(m-l)/(n-l)}$, is given in O'Malley [83].

Case 1: $\epsilon/\mu^{((m-l)/(n-l))} \to 0$ *as* $\mu \to 0$.

Introduce two new parameters $\sigma = \mu^{1/(n-l)}$ and

$$\nu = \left(\frac{\epsilon}{\mu}\right)^{1/(m-n)}, \quad \text{so that} \quad \frac{\nu}{\sigma} = \left(\frac{\epsilon}{\mu^{(m-l)/(n-l)}}\right)^{1/(m-n)} \to 0 \quad \text{as} \quad \mu \to 0.$$

For Case 1, then, $(n-l)$ singular solutions of (4.13) are of the form

$$D\left(x, \sigma, \left(\frac{\nu}{\sigma}\right)^{m-n}\right) = \frac{d\left(x, \sigma, \left(\frac{\nu}{\sigma}\right)^{m-n}\right)}{\sigma}$$

$$= \frac{1}{\sigma}\left[\omega_1{}^r g(x) + \sigma\left(\frac{\gamma_1(x) - \beta_1(x)}{n-l}\right)\right.$$

$$\left. - \left(\frac{\nu}{\sigma}\right)^{m-n}\left(\frac{(g(x)\,\omega_1{}^r)^{m-n+1}}{(n-l)\,\beta(x)}\right) + \cdots\right], \tag{4.14}$$

where $g(x)$ is positive, r is an integer, and ω_1 is a complex number of modulus one such that

$$(\omega_1{}^r g(x))^{n-l} = -\gamma(x)/\beta(x)$$

and each of the $(n-l)$ determinations $d(x, \sigma, (\nu/\sigma)^{m-n})$ can be expanded as a double power series in σ and $(\nu/\sigma)^{m-n}$ with variable coefficients which may be successively determined by substitution into (4.13). (For details, see O'Malley [81], Appendix A.) Let p_1 be the number of these roots whose real parts approach $-\infty$ as $\mu \to 0$ and let q_1 be the number of these roots whose real parts approach $+\infty$ as $\mu \to 0$. Further, $m-n$ singular roots of (4.13) are of the form

$$D\left(x, \nu, \left(\frac{\nu}{\sigma}\right)^{n-l}\right) = \frac{d\left(x, \nu, \left(\frac{\nu}{\sigma}\right)^{n-l}\right)}{\nu}$$

$$= \frac{1}{\nu}\left[\omega_2{}^s c(x) + \nu\left(\frac{\beta_1(x) - \alpha_1(x)}{m-n}\right)\right.$$

$$\left. + \left(\frac{\nu}{\sigma}\right)^{n-l}\left(\frac{\gamma(x)(\omega_2{}^s c(x))^{1-(n-l)}}{(m-n)\,\beta(x)}\right) + \cdots\right], \tag{4.15}$$

where $c(x)$ is positive, s is an integer, ω_2 is a complex number of modulus one such that

$$(\omega_2{}^s c(x))^{m-n} = -\beta(x)$$

and each $d(x, \nu, (\nu/\sigma)^{n-l})$ has a double series expansion in σ and $(\nu/\sigma)^{m-n}$ whose terms can be obtained successively. Define p_2 to be the number of these roots whose real parts approach $-\infty$ as $\mu \to 0$ and q_2 to be the number whose real parts approach $+\infty$.

Corresponding to each of these $m - l$ solutions of (4.13), it is frequently convenient to associate its singular part. That is, suppose there exists (least) positive integers K_1 and K_2 such that

$$\frac{1}{\sigma}\left(\frac{\nu}{\sigma}\right)^{K_1(m-n)} = o(1) = \frac{1}{\nu}\left(\frac{\nu}{\sigma}\right)^{K_2(n-l)} \tag{4.16}$$

as $\mu \to 0$. (Hereafter, we refer to these conditions as hypothesis (4.16).) Then, to $d(x, \sigma, (\nu/\sigma)^{m-n})$ and $d(x, \nu, (\nu/\sigma)^{n-l})$, we associate the corresponding finite expansions $\tilde{d}(x, \sigma, (\nu/\sigma)^{m-n})$ and $\tilde{d}(x, \nu, (\nu/\sigma)^{n-l})$ consisting of terms which make singular contributions in the expressions (4.14) and (4.15).

To each of the $n - l$ complex roots (4.14), we let correspond a formal complex solution of (4.11), namely,

$$y = \frac{A\left(x, \sigma, \left(\frac{\nu}{\sigma}\right)^{m-n}\right)}{(g(x))^{(n+l-1)/2}} \exp\left[\frac{1}{\sigma}\int_t^x d\left(s, \sigma, \left(\frac{\nu}{\sigma}\right)^{m-n}\right) ds\right], \tag{4.17}$$

and to the $m - n$ roots (4.15), the formal solutions

$$y = \frac{B\left(x, \nu, \left(\frac{\nu}{\sigma}\right)^{n-l}\right)}{(c(x))^{(m+n-1)/2}} \exp\left[\frac{1}{\nu}\int_t^x d\left(s, \nu, \left(\frac{\nu}{\sigma}\right)^{n-l}\right) ds\right]. \tag{4.18}$$

Moreover, when hypothesis (4.16) holds, we replace

$$d\left(s, \sigma, \left(\frac{\nu}{\sigma}\right)^{m-n}\right) \quad \text{and} \quad d\left(s, \nu, \left(\frac{\nu}{\sigma}\right)^{n-l}\right)$$

by the related finite sums

$$\tilde{d}\left(s, \sigma, \left(\frac{\nu}{\sigma}\right)^{m-n}\right) \quad \text{and} \quad \tilde{d}\left(s, \nu, \left(\frac{\nu}{\sigma}\right)^{n-l}\right).$$

In both cases, t is either 0 or 1, and A and B possess asymptotic double power series in σ and $(\nu/\sigma)^{m-n}$, and ν and $(\nu/\sigma)^{n-l}$, respectively, both with constant leading term and such that the variable higher order terms may be successively determined by Eq. (4.11) to within additive constants (see O'Malley [81], Appendix B).

Becoming more explicit, we define q_1 linearly independent solutions $y_1, y_2, \ldots, y_{q_1}$ of the equation (4.11) of the form (4.17) with $t = 1$

corresponding to the q_1 roots (4.14) whose real part approaches $+\infty$ as $\mu \to 0$, and to the p_1 roots whose real part approaches $-\infty$, we define linearly independent solutions $y_{q_1+1}, y_{q_1+2}, \dots, y_{q_1+p_1}$ of the form (4.17), all with $t = 0$. Likewise, to the q_2 roots (4.15) whose real part approaches $+\infty$ as $\mu \to 0$, we define linearly independent solutions $y_{n-l+1}, y_{n-l+2}, \dots, y_{n-l+q_2}$ of the form (4.18), all with $t = 1$, and to the p_2 roots whose real part approaches $-\infty$, we define linearly independent solutions $y_{n-l+q_2+1}, \dots, y_{n-l+q_2+p_2}$, all with $t = 0$. Clearly, $y_1, y_2, \dots, y_{q_1}$ and $y_{n-l+1}, y_{n-l+2}, \dots, y_{n-l+q_2}$ and their derivatives are all exponentially small away from $x = 1$, while $y_{q_1+1}, \dots, y_{q_1+p_1}$ and $y_{n-l+q_2+1}, \dots, y_{n-l+q_2+p_2}$ exhibit this boundary layer behavior near $x = 0$.

In order to simplify the exposition, we shall restrict attention here to the nonexceptional case where $p_1 + q_1 = n - l$ and $p_2 + q_2 = m - n$.

A remaining set of l linearly independent asymptotic solutions of (4.11) can be obtained as regular perturbations of any set $z_1, z_2, \dots, z_l$ of linearly independent solutions of the reduced equation $Lz = 0$, i.e., we set

$$y_{m-l+i} \sim \sum_{r,s=0}^{\infty} y_{irs}(x)\, \mu^r \epsilon^s, \qquad i = 1, 2, \dots, l,$$

where $y_{i00} = z_i$, and further terms are determined successsively from the differential equation

$$Ly_{irs} = -My_{i,r,s-1} - Ny_{i,r-1,s}$$

obtained by substituting the expansion for y_{m-l+i} into Eq. (4.11) and formally equating coefficients. (This nonhomogeneous equation can be solved by variation of parameters since the right-hand side is known at each step.) The results of Turrittin [102] show that $y_1, \dots, y_m$ form a fundamental system for (4.11).

Altogether then, we have constructed m linearly independent asymptotic solutions of Eq. (4.11) and therefore expect to write the limiting solution of the boundary value problem, if it exists, as a linear combination of this asymptotic fundamental system. Further, in event of convergence, we expect the limiting solution to converge uniformly in any closed interval outside the boundary layer to the solution of a reduced boundary value problem, consisting of the reduced equation $Lz = 0$ plus l boundary conditions. To determine which boundary conditions are canceled from the set (4.12), we must obtain a cancellation law.

Cancellation Law (Nonexceptional Case). *Cancel* $p = p_1 + p_2$ *boundary conditions at* $x = 0$ *and* $q = q_1 + q_2$ *boundary conditions at* $x = 1$, *starting from those involving the highest-order derivatives.*

Continuing in the manner of Wasow [109] as detailed in O'Malley [83], we obtain the following,

Theorem 4.5. *Consider the differential equation*

$$\epsilon My + \mu Ny + Ly = 0 \tag{4.11}$$

on the x-interval [0, 1] *subject to the boundary conditions*

$$y^{(\lambda_i)}(0) = l_i, \qquad i = 1, 2, ..., r,$$
$$y^{(\tau_i)}(1) = l_{r+i}, \qquad i = 1, 2, ..., m - r,$$

where

$$\beta(x) \neq 0 \neq \gamma(x), \qquad m > n > l \geqslant 0,$$

and

$$m > \lambda_1 > \lambda_2 > \cdots > \lambda_r \geqslant 0$$

and

$$m > \tau_1 > \tau_2 > \cdots > \tau_{m-r} \geqslant 0.$$

Further, let

$$\frac{\epsilon}{\mu^{(m-l)/(n-l)}} \to 0 \quad as \quad \mu \to 0,$$

let the differential equation be nonexceptional on the x-interval [0, 1], *and let* $z(x)$ *satisfy the reduced boundary value problem*

$$Lz = 0,$$
$$z^{(\lambda_i)}(0) = l_i, \qquad i = p + 1, p + 2, ..., r,$$
$$z^{(\tau_i)}(1) = l_{r+i}, \qquad i = q + 1, q + 2, ..., m - r,$$

where $p \leqslant r$ *and* $q \leqslant m - r$.

Suppose

1. *the reduced problem has a unique solution, and*
2. (a) $\lambda_1, \lambda_2, ..., \lambda_{p_2}$ *are distinct modulo* $m - n$,
   (b) $\lambda_{p_2+1}, \lambda_{p_2+2}, ..., \lambda_p$ *are distinct modulo* $n - l$,
   (c) $\tau_1, \tau_2, ..., \tau_{q_2}$ *are distinct modulo* $m - n$, *and*
   (d) $\tau_{q_2+1}, \tau_{q_2+2}, ..., \tau_q$ *are distinct modulo* $n - l$.

Then $y(x)$ has a well-defined limiting behavior on the interval $0 \leqslant x \leqslant 1$ as $\mu \to 0$ given by

$$y(x) = \frac{1}{(g(x))^{(n+l-1)/2}} \left[\sigma^{\tau_q} \sum_{k=1}^{q_1} u_k\left(x, \sigma, \frac{\nu}{\sigma}\right) + \sigma^{\lambda_p} \sum_{k=1}^{p_1} u_{k+q_1}\left(x, \sigma, \frac{\nu}{\sigma}\right)\right]$$

$$+ \frac{1}{(c(x))^{(m+n-1)/2}} \left[\sigma^{\tau_q} \left(\frac{\nu}{\sigma}\right)^{\tau_{q_2}} \sum_{k=1}^{q_2} u_{n-l+k}\left(x, \sigma, \frac{\nu}{\sigma}\right)\right.$$

$$\left. + \sigma^{\lambda_p} \left(\frac{\nu}{\sigma}\right)^{\lambda_{p_2}} \sum_{k=1}^{p_2} u_{n-l+q_2+k}\left(x, \sigma, \frac{\nu}{\sigma}\right)\right] + u\left(x, \sigma, \frac{\nu}{\sigma}\right), \tag{4.19}$$

where $u(x, \sigma, \nu/\sigma)$ is a regular perturbation of $z(x)$ in powers of σ and ν/σ, and

$$u_k\left(x, \sigma, \frac{\nu}{\sigma}\right) = \hat{A}_k\left(x, \sigma, \frac{\nu}{\sigma}\right) \exp\left[\frac{1}{\sigma} \int_t^x d\left(s, \sigma, \left(\frac{\nu}{\sigma}\right)^{m-n}\right) ds\right],$$

$$k = 1, 2, \ldots, n - l$$

and

$$u_k\left(x, \sigma, \frac{\nu}{\sigma}\right) = \hat{B}_k\left(x, \sigma, \frac{\nu}{\sigma}\right) \exp\left[\frac{1}{\nu} \int_t^x d\left(s, \nu, \left(\frac{\nu}{\sigma}\right)^{n-l}\right) ds\right]$$

$$k = n - l + 1, \ldots, m - l,$$

with the determinations of t and d (or $\bar{d}$) the same as that used in the expression for the corresponding y_k, and where $\hat{A}_k$ and $\hat{B}_k$ have asymptotic double power series in σ and ν/σ with constant leading term and higher-order terms determined uniquely from the differential equation and the boundary conditions. In particular, then,

$$y(x) \sim z(x) \quad on \quad 0 < x < 1.$$

Note. (1) That the hypotheses 1 and 2, and the conditions $p \leqslant r$, $q \leqslant m - r$ are essentially necessary is seen by considering a series of constant coefficient boundary value problems whose solutions, in general, diverge as $\epsilon, \mu \to 0$. See O'Malley [81] for a list of examples.

(2) If in the expansion for some $u_k(x, \sigma, \nu/\sigma)$,

$$d\left(s, \sigma, \left(\frac{\nu}{\sigma}\right)^{m-n}\right) \quad \text{or} \quad d\left(s, \nu, \left(\frac{\nu}{\sigma}\right)^{n-l}\right)$$

is represented by any finite partial sum $\hat{d}$ of its double series expansion, the resulting error in the function u_k has the same order as the difference $d - \hat{d}$.

Further results are reported in O'Malley [81]. There, e.g., the more complicated problem when equation (4.1) is exceptional is considered and a detailed determination of an asymptotic solution for the illustrative problem

$$\mu^2 y^{(7)} + \mu\beta(x)\, y^{(5)} + \gamma(x)\, y^{(2)} = 0$$

with

$$\beta(x) < 0 < \gamma(x) \qquad \text{on} \qquad [0, 1],$$

and

$$y^{(4)}(0), y^{(3)}(0), y^{(2)}(0), y'(0), y(0), y^{(2)}(1),$$

and

$$y'(1) \qquad \text{prescribed}$$

is given.

Case 2: $\mu^{((m-l)/(n-l))}/\epsilon \to 0$ *as* $\epsilon \to 0$.

In this case, we introduce $\kappa = \epsilon^{1/(m-l)} > 0$ [so that μ/κ^{n-l} is also a small parameter] and find $m - l$ singular roots of the auxiliary equation (4.13) of the form

$$\begin{aligned} D\left(x, \kappa, \frac{\mu}{\kappa^{n-l}}\right) &= \frac{d\left(x, \kappa, \frac{\mu}{\kappa^{n-l}}\right)}{\kappa} \\ &= \frac{1}{\kappa}\left[\omega^i h(x) + \kappa\left(\frac{\gamma_1(x) - \alpha_1(x)}{m - l}\right)\right. \\ &\quad \left. - \frac{\mu}{\kappa^{n-l}}\left(\frac{\beta(x)(g(x))^{1-(m-n)}}{m - l}\right) + \cdots\right], \end{aligned} \tag{4.20}$$

where $h(x)$ is positive, i is an integer, ω is a complex number of modulus one such that

$$(\omega^i h(x))^{m-l} = -\gamma(x),$$

and $d(x, \kappa, \mu/\kappa^{n-l})$ can be expanded as a double power series in κ and μ/κ^{n-l} such that each coefficient may be uniquely determined successively from Eq. (4.13). Further, to each of these $m - l$ singular roots, we associate formal solutions of (4.11), namely

$$y = \frac{E(x, \kappa, \mu/\kappa^{n-l})}{(h(x))^{(m+l-1)/2}} \exp\left[\frac{1}{\kappa}\int_t^x d(s, \kappa, \mu/\kappa^{n-l})\, ds\right],$$

where E has a double power series expansion in κ and μ/κ^{n-l} with constant leading term and higher-order terms obtainable successively from the differential equation (4.11) up to additive constants.

Again, we restrict attention to the nonexceptional case where $m - l = p + q$ for p the number of roots (4.20) whose real part approaches $-\infty$ as $\mu \to 0$ and q, the number whose real part approaches $+\infty$. We obtain the following results:

Cancellation Law (Nonexceptional Case). *Cancel p boundary conditions at $x = 0$ and q boundary conditions at $x = 1$, starting from those containing the highest derivatives.*

Theorem 4.6. *Consider the differential equation*

$$\epsilon My + \mu Ny + Ly = 0$$

on the x-interval $[0, 1]$ *subject to the boundary conditions*

$$y^{(\lambda_i)}(0) = l_i, \qquad i = 1, 2, \ldots, r,$$
$$y^{(\tau_i)}(1) = l_{r+i}, \qquad i = 1, 2, \ldots, m - r,$$

where

$$\gamma(x) \neq 0, \qquad m > n > l \geqslant 0,$$

and

$$m > \lambda_1 > \lambda_2 > \cdots > \lambda_r \geqslant 0,$$

and

$$m > \tau_1 > \tau_2 > \cdots > \tau_{m-r} \geqslant 0.$$

Further, let

$$\frac{\mu^{(m-l)/(n-l)}}{\epsilon} \to 0 \quad as \quad \epsilon \to 0,$$

let the differential equation be nonexceptional, and let $z(x)$ solve the reduced boundary value problem

$$Lz = 0,$$
$$z^{(\lambda_i)}(0) = l_i, \qquad i = p + 1, \ldots, r,$$
$$z^{(\tau_i)}(1) = l_{r+i}, \qquad i = q + 1, \ldots, m - r$$

where $p \leqslant r$ and $q \leqslant m - r$.

Suppose

1. *the reduced problem has a unique solution, and*
2. (a) $\lambda_1, \lambda_2, \ldots, \lambda_p$ *are distinct modulo* $m - l$,
   (b) $\tau_1, \tau_2, \ldots, \tau_q$ *are distinct modulo* $m - l$.

Then $y(x)$ *has a well-defined limiting behavior as* $\epsilon \to 0$ *given by*

$$y(x) = \frac{1}{(h(x))^{(m+l-1)/2}} \left[\kappa^{\tau_q} \sum_{k=1}^{q} u_k \left(x, \kappa, \frac{\mu}{\kappa^{n-l}}\right) + \kappa^{\lambda_p} \sum_{k=1}^{p} u_{q+k} \left(x, \kappa, \frac{\mu}{\kappa^{n-l}}\right)\right] + u(x, \kappa, \mu/\kappa^{n-l}), \tag{4.21}$$

where $u(x, \kappa, \mu/\kappa^{n-l})$ *is a regular perturbation of* $z(x)$, *and the linearly independent functions*

$$u_k \left(x, \kappa, \frac{\mu}{\kappa^{n-l}}\right) = A_k \left(x, \kappa, \frac{\mu}{\kappa^{n-l}}\right) \exp \left[\frac{1}{\kappa} \int_t^x d_k \left(s, \kappa, \frac{\mu}{\kappa^{n-l}}\right) ds\right],$$
$$k = 1, 2, \ldots, m - l$$

are determined formally, in the usual manner, where

$t = 1$ *and* $\operatorname{Re} d_k$ *has positive limiting values for* $k = 1, 2, \ldots, q$

while

$t = 0$ *and* $\operatorname{Re} d_k$ *has negative limiting values for* $k = q + 1, \ldots, m - l$.

Again,

$$y(x) \sim z(x) \quad \textit{on} \quad 0 < x < 1.$$

Note. (1) In the special (one-parameter) case where $\mu \equiv 0$, this theorem coincides with the conclusion in Wasow [109].

(2) For Case 2, the condition $\beta(x) \neq 0$ on $[0, 1]$ is not required in the nonexceptional case.

Comment. It seems feasible to also consider singular perturbation problems for systems of first-order equations involving many parameters. Thus, this author has analyzed systems of the form

$$\Omega(\epsilon)(dy/dx) = A(x, \epsilon)\, y \tag{4.19}$$

where $\Omega(\epsilon) = \operatorname{diag}(\epsilon_1 I_{m_1} : \epsilon_2 I_{m_2} : \cdots : \epsilon_s I_{m_s})$ (for I_{m_j}, the identity matrix of order m_j) and $A(x, \epsilon)$ is holomorphic in the small parameters ϵ_1/ϵ_2, $\epsilon_2/\epsilon_3, \ldots, \epsilon_{s-1}/\epsilon_s$, and ϵ_s. The system can (under appropriate conditions) be diagonalized in the manner of Wasow [115] and Harris [47] and the component problems can be analyzed by using the techniques of Turrittin [102] or Sibuya [96] (in the case of turning point problems).

Consider, e.g., boundary value problems for the equation

$$\epsilon z'' + \mu a(x) z' + b(x) z = 0$$

on the interval [0, 1] where $a(x) \neq 0$ when $\epsilon/\mu^2 \to 0$ as $\mu \to 0$. Introducing $y_1 = z'$ and $y_2 = z/\mu$, a system of the form (4.19) results with

$$\Omega(\epsilon) = \begin{pmatrix} \epsilon/\mu & 0 \\ 0 & \mu \end{pmatrix} \quad \text{and} \quad A(x, \epsilon) = \begin{pmatrix} -a(x) & -b(x) \\ 1 & 0 \end{pmatrix}.$$

Moreover, the results obtained above can be recovered using the plan outlined.

5. Relaxation Oscillations

The phenomenon known as relaxation oscillations (see Wasow [112]) and Friedman [33] arises in considering differential equations of the form

$$\epsilon \ddot{x} - F(\dot{x}) + x = 0 \tag{5.1}$$

or, equivalently, nonlinear autonomous systems

$$\begin{aligned} \dot{x} &= y, \\ \epsilon \dot{y} &= F(y) - x \end{aligned} \tag{5.2}$$

in the x–y phase plane for ϵ a small positive parameter. Differentiating (5.1) with respect to the independent variable t yields

$$\epsilon \ddot{y} - F'(y)\dot{y} + y = 0 \tag{5.3}$$

for $y = \dot{x}$. Note that (5.3) has a turning point (see, e.g., Wasow [116]) where $F'(y) = 0$. Moreover, in the special case where $F(y) = y - \frac{1}{3}y^3$, (5.3) is equivalent to the van der Pol equation

$$\frac{d^2y}{d\tau^2} - \lambda(1 - y^2)\frac{dy}{d\tau} + y = 0, \tag{5.4}$$

where $\lambda \equiv \epsilon^{-1/2}$ is a large positive parameter and $\tau \equiv \lambda t$, For $F(y) = y - \frac{1}{3}y^3$, define the "fundamental curve" $\Gamma : x = F(y)$ in the phase plane, noting that the reduced system ((5.2) with $\epsilon = 0$) is

$$\begin{cases} \dot{x} = y \\ 0 = F(y) - x. \end{cases}$$

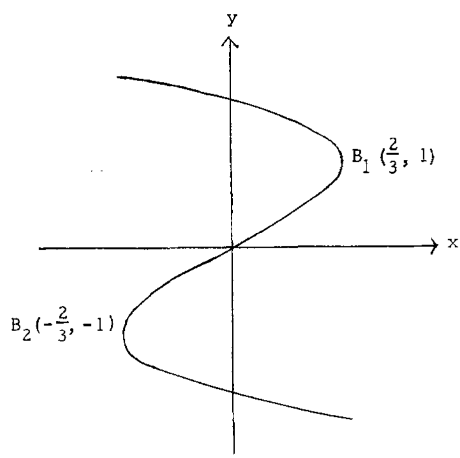

FIG. 1. The fundamental curve $\Gamma : x = F(y) = y - \frac{1}{3}y^3$.

The differential equation of the trajectories of (5.2) is

$$\frac{dy}{dx} = \frac{F(y) - x}{\epsilon y}. \tag{5.4}$$

Thus since this selection of $F(y)$ is an odd function, (5.4) implies that the vector field dy/dx will be symmetric with respect to the origin. Moreover, analyzing the field dy/dx determines the paths of (5.2). Specifically, let a trajectory start at a point P above the upper arc of Γ and to the left of $B_1 = (\frac{2}{3}, 1)$. As t increases, the trajectory must drop very rapidly, almost vertically, toward the fundamental curve Γ, then it will move to the right remaining close to, but above, Γ until it passes the point B_1, when it will drop rapidly, almost vertically, toward the lower arc of Γ. Crossing Γ, the trajectory will move to the left, remaining close to, but below, Γ until it passes $B_2 \equiv (-\frac{2}{3}, -1)$, when it will rise rapidly, almost vertically, to the upper arc of Γ. Crossing Γ, the trajectory will continue indefinitely in nearly the same orbit,

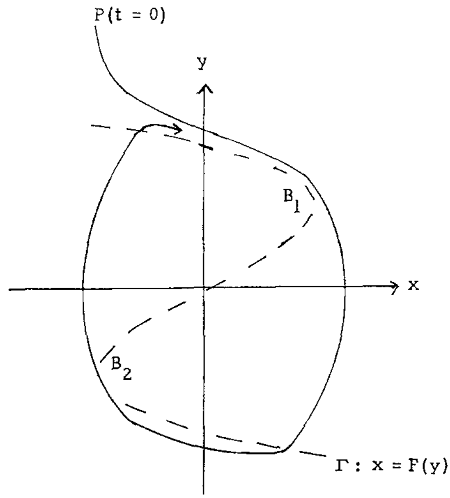

FIG. 2. Trajectory for (5.2) with $F(y) = y - \frac{1}{3}y^3$.

Liénard [69] showed that for each $\epsilon > 0$, (5.2) with $F(y) = y - \frac{1}{3}y^3$ has a unique (up to parameterization) limit cycle $C(\epsilon)$ (i.e., a periodic trajectory which other trajectories approach in a spiral-like manner as $t \to \infty$). That these limit cycles $C(\epsilon)$ actually have a limit as $\epsilon \to 0$ was proved in Flanders and Stoker [30]. This limit is the closed curve D formed by the outer arcs of the characteristic curve Γ and its two vertical tangents.

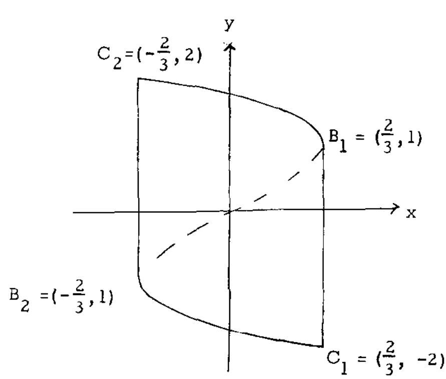

FIG. 3. The closed curve D.

Note that although D is a continuous curve in the phase plane, its tangent has discontinuities at $C_1 : (\frac{2}{3}, -2)$ and $C_2 : (-\frac{2}{3}, 2)$. Note, too, that (5.3), the scalar differential equation for y, has turning points at $y = \pm 1$, i.e., at B_1 and B_2. Further, only part of the limit curve D coincides with a path of a solution of the reduced system (5.2), and although D is a closed curve, this reduced system has no nontrivial periodic solutions and, hence, no closed trajectories.

Oscillations corresponding to the periodic trajectories of (5.3) are called relaxation oscillations and are characterized by jerky, almost instantaneous (in time, t), periodic jumps in $y = \dot{x}$. Expressing this physically, Stoker [98] states that these oscillations "exhibit two distinct and characteristic phases: one during which energy is stored up slowly (in a spring or condenser) and another in which the energy is discharged nearly instantaneously when a certain threshold potential is attained." This nonuniform convergence (in time) of the relaxation oscillations $C(\epsilon) : (x(t, \epsilon), y(t, \epsilon))$ of (5.2) as $\epsilon \to 0$ is, clearly, related to the boundary layer phenomenon discussed previously.

One expects that the limiting value of the period of the relaxation oscillations $C(\epsilon)$ as $\epsilon \to 0$ is given by

$$\lim_{\epsilon \to 0} \int_{C(\epsilon)} \frac{dx}{y} = 2 \int_0^2 \frac{dF(y)}{y} = 3 - 2 \log 2,$$

since $dt = dx/y$, $D = \lim_{\epsilon \to 0} C(\epsilon)$, $dx = 0$ on the vertical arcs of D, and $dx = dF(y)$ on Γ. Indeed, Dorodnicyn's analysis (see Dorodnicyn [23], Urabe [103], Urabe [104], and Ponzo and Wax [90]) gives the period $T(\epsilon)$ as

$$T(\epsilon) = (3 - 2 \log 2) + 3\alpha\epsilon^{2/3} + \tfrac{1}{3}\epsilon \log \epsilon + c\epsilon + O(\epsilon^{7/6})$$

where $-\alpha \doteq -2.338$ is the algebraically largest zero of the Airy function $Ai(z)$ and c is a given transcendental constant. Likewise, Dorodnicyn calculated the amplitude a of the y-oscillation, namely,

$$a = 2 + \frac{\alpha}{3}\epsilon^{2/3} + \frac{8}{27}\epsilon \log \epsilon + d\epsilon + O(\epsilon^{4/3}),$$

where d is another given transcendental constant. Roughly speaking, we could say that the "solution" D of the reduced system has y-amplitude 2 and "period" $3 - 2 \log 2$. Cartwright [13] and Haag [38], [39] both obtain similar results by different methods.

Stoker ([98], pp. 141–147) considers the illuminating, yet tractable problem of the relaxation oscillations for

$$\epsilon \ddot{x} - F(\dot{x}) + x = 0 \tag{5.1}$$

where F is the piecewise linear function

$$F(y) = \begin{cases} -2 - y, & y < -1, \\ y \quad , & -1 \leqslant y \leqslant 1, \\ 2 - y \; , & y > 1. \end{cases}$$

In this case, one must simply solve two linear, constant coefficient, second-order differential equations subject to matching and periodicity requirements. Proceeding in the obvious manner, the asymptotic solution for the relaxation oscillations is completely obtainable. In particular, the period $T(\epsilon)$ is given by

$$T(\epsilon) = 2[\log 3 - \tfrac{4}{3}\epsilon \log \epsilon + \epsilon(\tfrac{4}{3} - \log 3) + \tfrac{1}{9}\epsilon^2 \log^2 \epsilon] + O(\epsilon^2 \log \epsilon).$$

Comparing this expansion with the determination when $F(y) = y - \frac{1}{3}y^3$, is is clear that the form of the characteristic F markedly affects the nature of the asymptotic solution. In particular, we note that this polygonal characteristic F has discontinuous tangents at $y = \pm 1$.

These results were extended to more general fundamental curves by Mishchenko [75], and Mishchenko and Pontryagin [76]. Mishchenko considers the first-order system

$$\begin{cases} \dot{x} = f(x, y), \\ \epsilon \dot{y} = g(x, y), \end{cases}$$

where f and g are sufficiently smooth. Then if the reduced system has a "stable, discontinuous, periodic solution" $\mathfrak{F}_0$ with "period" T_0 (we shall define these terms precisely, below), the original (or full) system has a unique limit cycle $\mathfrak{F}_\epsilon$ near $\mathfrak{F}_0$, tending to $\mathfrak{F}_0$ as $\epsilon \to 0$, with period $T(\epsilon)$ given by the formula

$$T(\epsilon) = T_0 + \epsilon^{2/3} Q_1 + \epsilon \log \epsilon Q_2 + \epsilon Q_3 + O(\epsilon^{7/6})$$

for Q_1, Q_2, and Q_3 numerical constants depending only on the values of f, g, and their derivatives. [$\mathfrak{F}_0$ and T_0 are defined to include the van der Pol case.]

For a discussion of relaxation oscillation problems with time-dependent forcing term, see Cartwright [12], Wendell [117], and Brock [9].

THE TWO-TERM EXPANSION FOR THE AMPLITUDE AND PERIOD OF THE RELAXATION OSCILLATIONS FOR VAN DER POL'S EQUATION

Returning to van der Pol's equation (5.4), or equivalently the system (5.2) with $F(y) = y - \frac{1}{3}y^3$, we study the relaxation oscillation with periodic trajectory $C(\epsilon)$ passing through the points $A_i = (x_i, y_i)$, $i = 1, 2, \ldots, 7$.

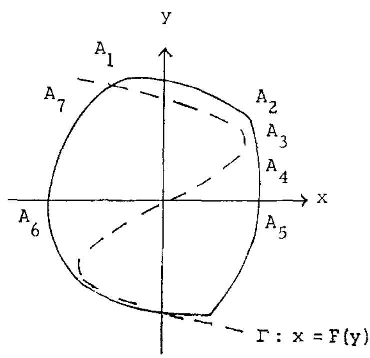

FIG. 4. Trajectory $C(\epsilon)$.

Let the trajectory $C(\epsilon)$ begin at some point A_1 on the fundamental curve $\Gamma : x = F(y)$, let $y_2 = 1 + \epsilon^r$ for $0 < r < \frac{1}{3}$, $y_3 = 1$, $y_4 = 1 - \epsilon^r$, $y_5 = y_6 = 0$, and $y_7 = y_1 - \epsilon$, and let $C(\epsilon)$ be symmetric with respect to the origin. Introduce

$$w = x - F(y) = x - y + \tfrac{1}{3}y^3,$$

the horizontal deviation of $C(\epsilon)$ from Γ, so that

$$\frac{dw}{dy} = -\frac{\epsilon y}{w} - F'(y) = \frac{y^2 - 1}{w}\left(w - \frac{\epsilon y}{y^2 - 1}\right).$$

Thus

$$0 < w(y) < \frac{\epsilon}{y - 1} \tag{5.5}$$

on $(1, y_1)$ (between A_1 and A_3) since either

$$w(y) \leqslant w(y^*) = \frac{\epsilon y^*}{(y^*)^2 - 1} < \frac{\epsilon}{y^* - 1}, \qquad y_1 > y^* > 1,$$

with y decreasing, or w is nonincreasing and

$$w(y) \leqslant \frac{\epsilon y}{y^2 - 1} < \frac{\epsilon}{y - 1}.$$

Near A_3, let

$$x = \tfrac{2}{3} + \epsilon^{2/3} u, \qquad y = 1 + \epsilon^{1/3} v$$

so that

$$0 \leqslant w(y) = \epsilon^{2/3} \left(u + v^2 + \epsilon^{1/3} \frac{v^3}{3} \right) < \frac{\epsilon^{2/3}}{v} \tag{5.6}$$

on $(1, y_1)$. Thus

$$v_2 = \epsilon^{r-1/3} \to \infty, \quad w(y_2) \to 0, \qquad \text{and} \qquad u_2 \to -\infty$$

as $\epsilon \to 0$. Further,

$$(u + v^2) \frac{du}{dv} + 1 = -\epsilon^{1/3} \left(\frac{v^3}{3} \frac{du}{dv} + v \right) \tag{5.7}$$

with the corresponding reduced equation

$$d\tilde{v}/du = -(\tilde{v}^2 + u) \tag{5.8}$$

having the general solution

$$\tilde{v}(u) = -\left(\frac{a_1 Ai'(-u) + a_2 Bi'(-u)}{a_1 Ai(-u) + a_2 Bi(-u)} \right)$$

where $Ai(z)$ and $Bi(z)$ are Airy functions and a_1 and a_2 are arbitrary constants (see Miller [73] and Copson [19] for the definitions and the relevant expansions of these functions). Assuming that the required solution v of (5.7) is a regular perturbation of some solution $\tilde{v}$ of (5.8), we seek a solution $\tilde{v}$ which approaches $+\infty$ as u approaches $-\infty$. Expanding $Ai(z)$ and $Bi(z)$ for large positive argument leads us to select $a_2 = 0$, so that $\tilde{v}(u) = -Ai'(-u)/Ai(-u)$. That $v = \tilde{v} + o(1)$ for $|v| \leqslant \epsilon^{r-1/3}$, r not too small, for this boundary condition, is shown in the appendix to Wasow [112], following the earlier argument by Haag. Further, $\tilde{v}'(u)$ is negative and uniformly bounded on $(-\infty, u_3)$ so that $u + \tilde{v}^2$ is bounded and

$$\tilde{v}(u) \leqslant \tilde{v}(u_2) \equiv v(u_2) = \epsilon^{r-1/3}$$

there. Thus, since $v = \tilde{v} + o(1)$, (5.6) implies that

$$w = O(\epsilon^{2/3})$$

on A_2A_3 for $\frac{2}{9} < r < \frac{1}{3}$.

Letting $y_4 = 1 - \epsilon^r$, $v_4 = -\epsilon^{r-1/3}$, so that $v_4 \to -\infty$ and u_4 must approach α as $\epsilon \to 0$ (using the expression for $\tilde{v}(u)$) where $-\alpha$ is the first zero of $Ai(z)$, i.e.,

$$x_4 = \tfrac{2}{3} + \epsilon^{2/3}\alpha + o(\epsilon^{2/3}).$$

Expanding about $(\frac{2}{3}, 1)$,

$$w = (y-1)^2 + \tfrac{1}{3}(y-1)^3 + (x - \tfrac{2}{3}) > (y-1)^2[1 - \tfrac{1}{3}(1-y)] > \tfrac{2}{3}(y-1)^2$$

on A_4A_5, so that

$$x_5 - x_4 = \epsilon \int_0^{y_4} \frac{y\,dy}{w} < \frac{3\epsilon}{2} \int_0^{1-\epsilon^r} (y-1)^{-2}\,dy$$
$$= O(\epsilon^{1-r}) = o(\epsilon^{2/3}).$$

Likewise,

$$\begin{aligned} -w &= -(x - x_1) - \tfrac{1}{3}(y-y_1)^3 - y_1(y-y_1)^2 + (1-y_1^2)(y-y_1) \\ &> -(y-y_1)[\tfrac{1}{3}(y-y_1)^2 + y_1(y-y_1) - (1-y_1^2)] \\ &> -(y-y_1)\left[\frac{y_1^3}{3} - 1\right] > -\delta(y-y_1) \end{aligned}$$

on A_6A_7 provided $y_1^2 > 3(1+\delta)$, $\delta > 0$. Thus,

$$x_7 - x_6 = \epsilon \int_0^{y_7} \frac{y\,dy}{-w} \leqslant -\frac{\epsilon y_1}{\delta} \int_0^{y_1-\epsilon} \frac{dy}{y-y_1} = O(\epsilon \log \epsilon).$$

Further, since

$$-w(dw/dy) = -w(y^2-1) + \epsilon y \geqslant \epsilon y$$

on A_7A_1,

$$w^2 \geqslant \epsilon(y_1^2 - y^2) > \epsilon y_7(y_1 - y),$$

and

$$x_1 - x_7 \leqslant \epsilon y_1 \int_{y_1-\epsilon}^{y_1} \frac{dy}{-w} \leqslant \frac{\epsilon^{1/2} y_1}{\sqrt{y_7}} \int_{y_1-\epsilon}^{y_1} \frac{dy}{\sqrt{y_1 - y}} = O(\epsilon).$$

Putting together these estimates,

$$x_1 = x_1 - x_7 + x_7 - x_6 - (x_5 - x_4) - x_4$$

since $x_6 = -x_5$, and therefore

$$x_1 = -\tfrac{2}{3} - \epsilon^{2/3}\alpha + o(\epsilon^{2/3}).$$

Further, since A_1 lies on Γ,

$$y_1 - 2 = \frac{-3(x_1 + \frac{2}{3})}{(y_1 + 1)^2} = \frac{\epsilon^{2/3}\alpha}{3} + o(\epsilon^{2/3}),$$

and the amplitude a of the relaxation oscillations for van der Pol's equation (5.4) is therefore

$$a = 2 + \tfrac{1}{3}(\epsilon^{2/3}\alpha) + o(\epsilon^{2/3})$$

as $\epsilon \to 0$ for $\alpha \doteq 2.338$ and $\epsilon = \lambda^{-2}$ since y_1 is the maximum value of y on the periodic trajectory $C(\epsilon)$.

Proceeding analogously, and using the preceding estimates, we can estimate the time T_{ij} required for traversing the arc A_iA_j. Specifically, we obtain the period T.

$$\begin{aligned} T &= 2(T_{12} + T_{23} + T_{34} + T_{45} + T_{67} + T_{71}) \\ &= 2\left[\left(\int_{y_1}^{y_3} \frac{dF(y)}{y} + o(\epsilon^{2/3})\right) + \epsilon^{2/3}(u_3 + o(1))\right. \\ &\qquad \left. + \epsilon^{2/3}(\alpha - u_3 + o(1)) + o(\epsilon^{2/3}) + O(\epsilon \log \epsilon) + O(\epsilon)\right] \\ &= 3 - 2\log 2 + 3\epsilon^{2/3}\alpha + o(\epsilon^{2/3}). \end{aligned}$$

6. Initial-Value Problems for Nonlinear Systems of Equations

Perturbations of Discontinuous Solutions

Levinson [65], [67] considers the nonlinear system

$$\begin{cases} \dot{x} = f\dot{y} + r, \\ \epsilon\ddot{y} + g\dot{y} + h = 0 \end{cases} \tag{6.1}$$

for x an n-vector, y a scalar, and ϵ a small, positive parameter where the vectors f and r and the scalars g and h are functions of x, y, and the independent variable t.

Note that this general system includes systems of the form

$$\frac{dx}{dt} = f(x, y, t),$$

$$\epsilon \frac{dy}{dt} = g(x, y, t)$$

[by differentiating the last equation, or by introducing $w = \int^t y(s)\, ds$], and relaxation oscillation problems given by scalar equations of the form

$$\epsilon \ddot{x} - F(\dot{x}) + x = 0$$

(by differentiating and introducing $y = \dot{x}$).

Corresponding to (6.1), we define the reduced (or degenerate) system

$$\begin{cases} \dot{u} = f\dot{v} + r, \\ g\dot{v} + h = 0. \end{cases} \tag{6.2}$$

Note that dv/dt may become singular when g vanishes.

Recall the special case of relaxation oscillations where the limit of the phase plane limit cycles consists of "regular arcs" on which the reduced system is satisfied and of "jump arcs" "traversed in zero time" (i.e., nearby trajectories which are traversed nearly instantaneously). For the more complicated examples encompassed by (6.1), however, the path of the degenerate trajectory in E_{n+1} (i.e., the path of the limiting solution, if its exists) is not *à priori* clear. Levinson defines the solution of the reduced system (6.2) as a curve C_0 in E_{n+2} with coordinates (u, v, t) in such a manner that these questions are clarified and most potent results are obtained.

The vector $(u(t), v(t), t) \in E_{n+2}$, $0 \leqslant t \leqslant 1$, is defined to be a solution of the reduced system (6.2) if

(1) $(u(t), v(t))$ is continuous for $0 \leqslant t \leqslant 1$ except for a finite number of values t_j, $0 < t_1 < t_2 < \cdots < t_m < 1$. Moreover, the limits $(u(t_j \pm 0), v(t_j \pm 0))$ exist for each j, $j = 1, 2, \ldots, m$.

(2) $(u(t), v(t))$ satisfies (6.2) except for $t = t_j$, $j = 1, 2, \ldots, m$.

(3) $g(u(t), v(t), t)$ is positive for all $t \in [0, 1]$ except possibly for the values $t = t_j$. Moreover, for each j, $g(u(t), v(t), t)$ tends to zero as $t \to t_j^-$ while g remains positive as $t \to t_j^+$.

(4) At each point,

$$(u_-, v_-, t_j) \equiv (u(t_j - 0), v(t_j - 0), t_j)),$$

we require that

$$I \equiv \sum_{i=1}^{n} \frac{\partial g}{\partial u_i} f_i + \frac{\partial g}{\partial v} \neq 0$$

and further that

$$h_- \equiv h(u_-, v_-, t_j) \neq 0.$$

(5) From each coordinate u_-, u satisfies the initial-value problem

$$\frac{du}{dv} = f(u, v, t_j)$$

$$u(v_-) = u_-,$$

where v increases if h_- is negative and decreases otherwise, until the first value of $v \neq v_-$ is reached for which

$$\int_{v_-}^{v} g(u(v), v, t_j)\, dv = 0.$$

This value of v is denoted by $v(t_j + 0)$ and the associated value of u by $u(t_j + 0)$.

Thus C_0, as constructed, is continuous as a curve in E_{n+1} (but not as a function of t), but it has a discontinuous tangent, in general, where $t = t_j$, $j = 1, 2, \ldots, m$.

Note. (1) The construction of the solution C_0 of (6.2) places specific requirements on the functions $f, r, g,$ and h. These conditions are sufficient for the theorems which follow. They are not necessary, e.g., Levinson states that if I changes sign where g becomes zero, no jumps will occur.

(2) In relaxation oscillation problems, the solution (u, v) of the reduced system (6.2) features jumps in the scalar v only, since the function f is then identically zero.

With this definition of C_0, Levinson proves the following.

Theorem 6.1. *Consider the nonlinear system*

$$\dot{x} = f\dot{y} + r,$$

$$\epsilon\ddot{y} + g\dot{y} + h = 0,$$

where x is an n-vector and y is a scalar with $x(0)$ and $y(0)$ prescribed. Further, let $(u(t), v(t))$ solve the corresponding reduced system

$$\dot{u} = f\dot{v} + r,$$

$$g\dot{v} + h = 0,$$

for $t \in [0, 1]$ with initial conditions $(u(0), v(0))$ such that

$$\| x(0) - u(0) \| + | y(0) - v(0) | \leqslant \delta_1$$

and

$$\left| \frac{dy}{dt}(0) - \frac{dv}{dt}(0) \right| \leqslant \frac{\delta_2}{\epsilon}.$$

Then the full system has a solution $(x(t), y(t))$ for $t \in [0, 1]$ provided ϵ, δ_1, and δ_2 are all sufficiently small. Moreover, as ϵ, δ_1, and δ_2 tend to zero, the solution curve $(x(t), y(t), t)$ in E_{n+2} tends to $C_0 : (u(t), v(t), t)$. In particular, for any fixed $\delta > 0$,

$$\| x(t) - u(t) \|, \qquad | y(t) - v(t) |,$$

and

$$\left| \frac{dy}{dt} - \frac{dv}{dt} \right|$$

all tend uniformly to zero over the intervals

$$0 < \delta \leqslant t \leqslant t_1 - \delta, \qquad t_1 + \delta \leqslant t \leqslant t_2 - \delta, \ldots,$$

$$t_{m-1} + \delta \leqslant t \leqslant t_m - \delta, \qquad t_m + \delta \leqslant t \leqslant 1 - \delta$$

as ϵ, δ_1, and δ_2 tend to zero.

The existence of a solution C_0, even though discontinuous, of the reduced system (6.2), then, implies the existence of nearby solutions of the full system (6.1). Additional theorems discuss the analogous convergence of the derivatives of $x(t)$ and $y(t)$ with respect to initial conditions. In particular, when f, r, g, and h of (6.1) do not depend directly on t or in case they are periodic in t of period $\tilde{T}$, if the reduced system (6.2) has a solution of period $\tilde{T}$ and if the Jacobian associated with the determination of this solution by varying initial conditions is different from zero, then the full system (6.1) will also have a nearby periodic solution, say of period $T \sim \tilde{T}$ as $\epsilon, \delta_1, \delta_2 \to 0$. For then, the equations

$$\varphi_1(T) \equiv x(T) - x(0) = 0,$$
$$\varphi_2(T) \equiv y(T) - y(0) = 0,$$
$$\varphi_3(T) = \frac{\partial y}{\partial t}(T) - \frac{\partial y}{\partial t}(0) = 0$$

must hold for small ϵ, δ_1, and δ_2 which requires that the Jacobian

$$\frac{\partial(\varphi_1(T), \varphi_2(T), \varphi_3(T))}{\partial\left(x(0), y(0), \dfrac{\partial y}{\partial t}(0)\right)}$$

be continuous and nonvanishing as ϵ, δ_1, and $\delta_2 \to 0$. The convergence results, however, show that this will be satisfied provided only that the Jacobian

$$\frac{\partial(\psi_1(\tilde{T}), \psi_2(\tilde{T}))}{\partial(u(0), v(0))}$$

is nonvanishing for $\psi_1(\tilde{T}) = u(\tilde{T}) - u(0)$ and $\psi_2(\tilde{T}) = v(\tilde{T}) - v(0)$.

Asymptotic Solutions for Initial-Value Problems

Vasil'eva (see, especially the survey article, Vasil'eva [106], where the list of her previous papers can be found) considers the system of nonlinear equations

$$\begin{cases} dx/dt = f(x, y, t), \\ \epsilon(dy/dt) = g(x, y, t), \end{cases} \tag{6.3}$$

for ϵ a small positive parameter, x and y vectors, and f and g sufficiently smooth functions. Introducing

$$z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

let

$$z(0) = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$$

be prescribed.

Since the reduced system

$$\begin{cases} dx/dt = f(x, y, t), \\ \quad 0 = g(x, y, t), \end{cases} \tag{6.4}$$

corresponding to (6.3) is of lower order than the full system (6.3), it cannot, in general, satisfy all the initial conditions prescribed for $z(t)$. Obvious questions therefore arise. Does the solution of the initial value problem (6.3) have a well-defined limiting solution as $\epsilon \to 0$? If so, does this limiting solution satisfy the reduced system? In that case, does the limiting solution assume the initial condition $x(0)$, and further, which solution of the system $g(x, y, t) = 0$ determines the limiting behavior of y? Before analyzing these questions, it is necessary to introduce further terminology.

Let $y = \varphi(x, t)$ be any solution of the system of equations $g(x, y, t) = 0$ defined on a closed bounded set D. We shall denote the solution of the reduced system (6.4) corresponding to the root $y = \varphi(x, t)$ with $x(0)$ prescribed by

$$\bar{z}(t) = \begin{pmatrix} \bar{x}(t) \\ \varphi(\bar{x}(t), t) \end{pmatrix}.$$

Further, we shall call the root $y = \varphi(x, t)$ isolated on D if there exists some positive number γ such that $g(x, y, t) = 0$ has no root other than $y = \varphi(x, t)$ for $|y - \varphi(x, t)| < \gamma$.

For the full system (6.3), y is rapidly-varying compared to x and t when ϵ is small and g is positive. This observation leads us to consider the boundary layer (or adjoined) system

$$dy/d\tau = g(x^*, y, t^*), \tag{6.5}$$

where $\tau = t/\epsilon$ and x^* and t^* are fixed parameters.

An isolated root $y = \varphi(x, t)$ will be called stable in D if, for all points $(x^*, y^*) \in D$, the points $y^* = \varphi(x^*, t^*)$ are asymptotically stable critical points of the boundary layer equation (6.5) as $\tau \to \infty$, uniformly for $(x^*, y^*) \in D$; i.e.:

1. $g(x^*, y^*, t^*) = 0$;

2. for each $\mu > 0$, there exists $\delta(\mu) > 0$ independent of x^* and t^* such that

$$\| y(0) - y^* \| < \delta(\mu)$$

implies that a solution $y(\tau)$ of (6.5) exists and satisfies the inequality

$$\| y(\tau) - y^* \| < \mu$$

for all $\tau > 0$ as well as the asymptotic requirement

$$\lim_{\tau \to \infty} y(\tau) = y^* = \varphi(x^*, t^*).$$

(For an elementary introduction to stability theory, see LaSalle and Lefschetz [62].)

Lastly, a point (x^*, y^*, t^*) is said to be in the domain of influence of the stable root $y = \varphi(x, t)$ if the solution of the boundary layer system (6.5) satisfying the initial condition $y\,|_{\tau=0} = y^*$ tends to the value $\varphi(x^*, t^*)$ as $\tau \to \infty$. [Here, $y^* \neq \varphi(x^*, t^*)$, in general.] Using this terminology, we are able to state the basic conclusion of Tikhonov [99]:

Theorem 6.2. *Suppose*

(1) *the root $y = \varphi(x, t)$ of the system of equations $g(x, y, t) = 0$ is continuous, isolated, and stable in some closed, bounded domain D,*

(2) *the point $(x(0), y(0), 0)$ belongs to the domain of influence of this root,*
and

(3) *the solution $\bar{x}(t)$ of the reduced initial-value problem*

$$\begin{aligned} dx/dt &= f(x, y, t), \\ y &= \varphi(x, t), \\ x(0) &\quad \textit{prescribed} \end{aligned}$$

is unique and lies in D for $0 \leqslant t \leqslant T$.
Then the solution of the initial-value problem for the full system

$$\begin{aligned} dx/dt &= f(x, y, t) \\ \epsilon(dy/dt) &= g(x, y, t) \\ x(0),\ \ y(0) &\quad \textit{prescribed} \end{aligned}$$

converges to the solution $(\bar{x}(t), \bar{y}(t))$ of the reduced system as $\epsilon \to 0$ in such a manner that

$$\lim_{\epsilon \to 0} x(t) = \bar{x}(t), \qquad 0 \leqslant t \leqslant T_0 < T,$$

$$\lim_{\epsilon \to 0} y(t) = \bar{y}(t) = \varphi(\bar{x}(t), t), \qquad 0 < t \leqslant T_0 < T$$

where T_0 is arbitrarily close to, but less than, T.

Note. (1) As Wasow [116] points out, this formulation of the Tikhonov theorem implicitly assumes the existence of a unique solution of the full initial-value problem on $0 \leqslant t \leqslant T$. Note, however, that since we have asked that f and g be sufficiently smooth, the existence is assured.

(2) The limiting solution of the full initial-value problem is discontinuous at $t = 0$ unless, by chance, $y(0) = \varphi(x(0), 0)$. In a small neighborhood of $t = 0$, then, a boundary layer phenomenon occurs as the solution passes rapidly from the initial point of the full system $(x(0), y(0), 0)$ into the neighborhood of the initial point $(x(0), \varphi(x, 0), 0)$ of the reduced system.

(3) In Tikhonov's definition of a stable root $y = \varphi(x, t)$, the asymptotic stability is not required to be uniform for $(x^*, y^*) \in D$. That this is incorrect follows by a counterexample given in Hoppensteadt [49].

(4) The analogous theorem for t lying in the semi-infinite interval $[0, \infty)$ is proved in Hoppensteadt [48]. In particular, this paper contains a new proof of the Tikhonov theorem. These results may also be extended to initial-value problems involving several parameters.

(5) Murphy [78] reports an interesting method for solving initial-value problems of the form (6.3) numerically.

An Illustrative Example

Consider the nonlinear autonomous system

$$\frac{dx}{dt} = e^{x^3} \sin y + e^{xy^2} \sin x - y^2 - xy - 1,$$

$$\epsilon(dy/dt) = y^2 - x^2,$$

with the initial conditions $x(0) = \alpha$ and $y(0) = \beta$ prescribed. Tikhonov's theorem implies that the solution of this problem converges (as $\epsilon \to 0$) to the solution of the reduced initial-value problem

$$dx/dt = -1,$$

$$y = -x, \qquad x(0) = \alpha,$$

provided $\alpha > 0$ for all values of t on any closed subinterval of $(0, \alpha)$.

For $\alpha < 0$, however, the solution converges to the solution of the nonlinear reduced initial-value problem

$$\frac{dx}{dt} = 2(e^{x^3} \sin x - x^2) - 1,$$

$$v = x, \qquad x(0) = \alpha,$$

for t on any closed (and bounded) subinterval of $(0, \infty)$.

For this example, $g(x, y, t) \equiv y^2 - x^2$ so that the roots of the equation $g(x, y, t) = 0$ are $y = \pm x$. Each of these roots is isolated in any closed, bounded subset D of the $x - y$ plane which does not intersect the line $x = 0$.

The boundary layer equation

$$dy/d\tau = y^2 - x^{*2}$$

where $\tau = t/\epsilon$ and x^* is a fixed parameter has the general solution

$$y(\tau) = -x^* \left(\frac{a_1 e^{x^* \tau} + a_2 e^{-x^* \tau}}{a_1 e^{x^* \tau} - a_2 e^{-x^* \tau}} \right)$$

for a_1 and a_2 arbitrary constants, while the initial-value problem is solved by

$$y(\tau) = -x^* \left[\frac{(y(0) - x^*)\, e^{x^* \tau} + (y(0) + x^*)\, e^{-x^* \tau}}{(y(0) - x^*)\, e^{x^* \tau} - (y(0) + x^*)\, e^{-x^* \tau}} \right].$$

For x^* lying in any closed, bounded subset D_1 of the right half-plane (we take the half-planes to be open sets),

$$| y(0) - (-x^*)| < \tfrac{1}{2}\mu$$

implies that $y(\tau)$ exists for all $\tau > 0$ and that

$$| y(\tau) - (-x^*)| = \left| \frac{2x^*(y(0) + x^*)\, e^{-2x^* \tau}}{(y(0) - x^*) - (y(0) + x^*)\, e^{-2x^* \tau}} \right| < \mu$$

for μ sufficiently small. In D_1, then, $y(\tau) \to -x^*$, so that the root $y = -x$ is stable and the root $y = x$ is not stable. Analogously, for any closed, bounded subset D_2 of the left half-plane, the root $y = x$ is stable while the root $y = -x$ is not stable.

Moreover, any point in the right half-plane lies in the domain of influence of the root $y = -x$, and any point in the left half-plane is in the domain of influence of the root $y = x$.

If $x(0) = \alpha > 0$, the solution of the initial-value problem

$$dx/dt = -1, \qquad x(0) = \alpha$$

will remain positive for $t \in [0, \alpha)$. Likewise, the solution of the initial-value problem

$$\frac{dx}{dt} = 2(e^{x^3} \sin x - x^2) - 1, \qquad x(0) = \alpha$$

for $\alpha < 0$ will remain negative as t increases since $2(e^{x^3} \sin x - x^2) - 1 < 0$ for $x < 0$. Applying Theorem 6.2, then, the stated results follow immediately.

The Construction Procedure

To construct the asymptotic solutions desired, Vasil'eva asks somewhat more than the requirement that the root $y = \varphi(x, t)$ of the system $g(x, y, t) = 0$ be uniformly asymptotically stable. Instead, she asks that the real parts of the characteristic roots of the matrix

$$(\partial g/\partial y)(x, \varphi(x, t), t) \tag{6.6}$$

be negative in D. Hereafter, we refer to this condition as hypothesis (6.6).

Following Vasil'eva, we first construct an "inner" solution of the initial-value problem valid in the boundary layer near $t = 0$. The initial-value problem for (6.3) is equivalent to the initial-value problem for the system

$$\begin{cases} dx/d\tau = \epsilon f(x, y, \epsilon\tau), \\ dy/d\tau = g(x, y, \epsilon\tau) \end{cases} \tag{6.7}$$

for $\tau = t/\epsilon$. Proceeding, we let $z(\tau) = \binom{x(\tau)}{y(\tau)}$ and set

$$z(\tau) = \sum_{k=0}^{\infty} z_k(\tau)\, \epsilon^k$$

(with $z(0)$ prescribed),

$$f(x(\tau), y(\tau), \epsilon\tau) = \sum_{k=0}^{\infty} \epsilon^k f_k(\tau),$$

and

$$g(x(\tau), y(\tau), \epsilon\tau) = \sum_{k=0}^{\infty} \epsilon^k g_k(\tau).$$

Equating coefficients of each power of ϵ in (6.7) to zero yields

$$\begin{cases} dx_k/d\tau = f_{k-1}(\tau), \\ dy_k/d\tau = g_k(\tau), \end{cases} \tag{6.8}$$

for each $k \geqslant 0$ where $f_{-1}(\tau) \equiv 0$, $z_0(0) = z(0)$, and $z_k(0) = 0$ for $k > 0$. Integrating then, for $k = 0$,

$$x_0(\tau) = x(0)$$

$$\frac{dy_0}{d\tau} = g(x_0(\tau), y_0(\tau), 0), \qquad y_0(0) = y(0). \tag{6.9a}$$

By the hypotheses of Theorem 6.2, however, we are guaranteed that (6.9a) has a solution which converges to $\varphi(x(0), 0)$ as $\tau \to \infty$ (since (6.9a) coincides with the boundary layer equation). Likewise,

$$dx_1/d\tau = f(x_0(\tau), y_0(\tau), 0), \qquad x_1(0) = 0$$

implies that

$$x_1(\tau) = \int_0^\tau f(x_0(s), y_0(s), 0)\, ds$$

and

$$\begin{aligned} \frac{dy_1}{d\tau} &= g_x(x_0(\tau), y_0(\tau), 0)\, x_1(\tau) \\ &\quad + g_y(x_0(\tau), y_0(\tau), 0)\, y_1(\tau) \\ &\quad + g_t(x_0(\tau), y_0(\tau), 0)\, \tau \\ &\equiv g_y(x_0(\tau), y_0(\tau), 0)\, y_1(\tau) + \tilde{q}_1(\tau), \end{aligned} \tag{6.8b}$$

with $y_1(0) = 0$ can also be integrated. In general, for $k > 1$,

$$\begin{aligned} f_k(\tau) &= f_x(x_0(\tau), y_0(\tau), 0)\, x_k(\tau) \\ &\quad + f_y(x_0(\tau), y_0(\tau), 0)\, y_k(\tau) + p_{k-1}(\tau) \end{aligned}$$

and

$$\begin{aligned} g_k(x) &= g_x(x_0(\tau), y_0(\tau), 0)\, x_k(x) \\ &\quad + g_y(x_0(\tau), y_0(\tau), 0)\, y_k(x) + q_{k-1}(\tau), \end{aligned}$$

where the $p_{k-1}(\tau)$ and $q_{k-1}(\tau)$ depend on τ, and the $x_j(\tau)$ and $y_j(\tau)$ for $j \leqslant k - 1$. Continuing, then, the $x_k(\tau)$ and $y_k(\tau)$ may be uniquely obtained successively, in order, from the linear equations (6.8) and the condition that $z_k(0) = 0$ for $k > 0$.

By Tikhonov's theorem, however, the solution of the initial-value problem for (6.3) is asymptotically equal to a solution of the reduced problem for (6.4) with $x(0)$ prescribed outside the boundary layer. Thus, following Vasil'eva, we develop $z(t)$ in an "outer solution" about $\epsilon = 0$, i.e., we let

$$\bar{z}(t) = \sum_{k=0}^{\infty} \bar{z}_k(t)\, \epsilon^k,$$

$$f(\bar{x}(t), \bar{y}(t), t) = \sum_{k=0}^{\infty} \bar{f}_k(t)\, \epsilon^k,$$

and

$$g(\bar{x}(t), \bar{y}(t), t) = \sum_{k=0}^{\infty} \bar{g}_k(t)\, \epsilon^k.$$

(*Note.* $\bar{x}(t)$ and $\bar{y}(t)$ are not the same here as the functions so denoted in the statement of Theorem 6.1.) Then, formally equating powers of ϵ in (6.3), we obtain

$$\begin{cases} d\bar{x}_k/dt = \bar{f}_k(t), \\ \bar{g}_k(t) = d\bar{y}_{k-1}/dt \end{cases} \tag{6.10}$$

for $k \geqslant 0$ where $\bar{y}_{-1} \equiv 0$. In particular, when $k = 0$,

$$\bar{g}_0(t) \equiv g(\bar{x}_0(t), \bar{y}_0(t), t) = 0$$

and we select $\bar{y}_0(t) = \varphi(\bar{x}_0(t), t)$ which, in turn, requires that $\bar{x}_0$ must satisfy the reduced system corresponding to the root $\varphi(x, t)$, i.e.,

$$dx/dt = f(x, y, t),$$

$$y = \varphi(x, t),$$

with $x(0)$ prescribed. The hypotheses of Theorem 6.2, however, assume that this nonlinear problem is solvable for $0 \leqslant t \leqslant T$. For $k = 1$,

$$d\bar{x}_1/dt = f_x(\bar{x}_0(t), \bar{y}_0(t), t)\, \bar{x}_1(t) + f_y(\bar{x}_0(t), \bar{y}_0(t), t)\, \bar{y}_1(t)$$

and

$$\bar{g}_1(t) = g_x(\bar{x}_0(t), \bar{y}_0(t), t)\, \bar{x}_1(t) + g_y(\bar{x}_0(t), \bar{y}_0(t), t)\, \bar{y}_1(t) = d\bar{y}_0/dt.$$

Since $g_y(\bar{x}_0(t), \bar{y}_0(t), t)$ is invertible, however, $\bar{y}_1(t)$ is a well-defined function of $\bar{x}_1(t)$, and $\bar{x}_1(t)$ satisfies a simple, linear, nonhomogeneous system of equations. Thus $\bar{x}_1(t)$ and $\bar{y}_1(t)$ are completely determined on $0 \leqslant t \leqslant T$ up to the selection of the initial condition $\bar{x}_1(0)$. (Note that Tikhonov's theorem does not yield higher-order approximate solutions, and therefore gives no information regarding $\bar{x}_k(0)$ for $k > 0$.) Likewise, for $k > 1$,

$$\bar{f}_k(t) = f_x(\bar{x}_0(t), \bar{y}_0(t), t)\, \bar{x}_k + f_y(\bar{x}_0(t), \bar{y}_0(t), t)\, \bar{y}_k + \bar{p}_{k-1}(t)$$

and

$$\bar{g}_k(t) = g_x(\bar{x}_0(t), \bar{y}_0(t), t)\, \bar{x}_k + g_y(\bar{x}_0(t), \bar{y}_0(t), t)\, \bar{y}_k + \bar{q}_{k-1}(t),$$

where $\bar{p}_{k-1}(t)$ and $\bar{q}_{k-1}(t)$ depend on t and $\bar{x}_j(t)$ and $\bar{y}_j(t)$ for $j < k$. Solving for $\bar{y}_k(t)$ in terms of $\bar{x}_k(t)$ and integrating, the $\bar{x}_k(t)$ and $\bar{y}_k(t)$ can be successively determined on the interval $[0, T]$ up to the selection of $\bar{x}_k(0)$.

Intuitively, we expect to obtain these unknown constants by, somehow, matching the inner expansion $z(\tau)$ (which can be completely determined to any number of terms) with the outer expansion $\bar{z}(t)$ (which can be determined up to these constants through any number of terms) for large values of τ and small values of t. Thus, we formally expand the outer solution $\bar{z}(t)$ about $t = 0$, i.e., we set

$$\tilde{z} = \sum_{m=0}^{\infty} \sum_{j=0}^{m} z_{j,m-j} t^j \epsilon^{m-j}$$

where the z_{jl}'s are defined by the expansions

$$\bar{z}_l(t) = \sum_{j=0}^{\infty} z_{jl} t^j.$$

Further, we let $(\tilde{z})_N$ represent the partial sum of $\tilde{z}$ through all Nth-order terms, in the two variables t and ϵ. Expressing $(\tilde{z})_N$ in terms of $\tau = t/\epsilon$, we have

$$(\tilde{z})_N = \sum_{m=0}^{N} \tilde{z}_m(\tau)\, \epsilon^m$$

for

$$\tilde{z}_m(\tau) = \sum_{j=0}^{m} z_{j,m-j}\tau^j.$$

Letting $(z)_N$ be the N-term inner expansion, the difference $(z)_N - (\tilde{z})_N$ satisfies

$$(z)_N - (\tilde{z})_N = \sum_{m=0}^{N} \prod\nolimits_m (z, \tau)\, \epsilon^m$$

for the "boundary layer functions"

$$\prod\nolimits_m (z, \tau) = z_m(\tau) - \tilde{z}_m(\tau).$$

Note that

$$\prod\nolimits_m (z, 0) = \begin{cases} 0 \quad , & m = 0, \\ -\tilde{z}_m(0), & m > 0. \end{cases}$$

Since

$$(d/d\tau) \prod\nolimits_m (x, \tau) = f_{m-1}(\tau) - \tilde{f}_{m-1}(\tau)$$

where

$$\tilde{f}_{m-1}(\tau) = \sum_{i=0}^{n-1} f_{i,m-1-i}\tau^i$$

and

$$f_k(\tau) = \sum_{l=0}^{\infty} f_{lk}t^l,$$

$$\prod\nolimits_m (x, \tau) = \left[\prod\nolimits_m (x, 0) + \int_0^\infty [f_{m-1}(\tau) - \tilde{f}_{m-1}(\tau)]\, d\tau\right] - \int_\tau^\infty [f_{m-1}(s) - \tilde{f}_{m-1}(s)]\, ds.$$

Thus $\prod_m(x, \tau) \to 0$ as $\tau \to \infty$ (i.e., at the matching point) provided

$$\bar{x}_m(0) = \int_0^\infty [f_{m-1}(\tau) - \tilde{f}_{m-1}(\tau)]\, d\tau \tag{6.11}$$

for each $m > 0$.

In general, we expect that $(z)_N - (\tilde{z})_N$ is negligible outside the boundary layer, while $(\bar{z})_N - (\tilde{z})_N$ for $(\bar{z})_N = \sum_{k=0}^{N} \bar{z}_k(t)\epsilon^k$ is negligible within the boundary layer. Further, we expect that the solution z of the initial-value problem will have the asymptotic solution $z(\tau)$ within the boundary layer and $\tilde{z}(t)$ outside the boundary layer. Expressing this formally, Vasil'eva proves

Theorem 6.3. *Let $z(t, \epsilon)$ solve the nonlinear initial-value problem for the system of equations* (6.3). *Further, let the hypotheses of Theorem* 6.2 *hold as well as hypothesis* (6.6), *and let the expansions $(\bar{z})_N$, $(z)_N$, and $(\tilde{z})_N$ be those formally constructed above. Then for each integer $N \geqslant 0$,*

$$z(t, \epsilon) = Z_N + O(\epsilon^{N+1})$$

uniformly on $0 \leqslant t \leqslant T_0$ for all sufficiently small ϵ where

$$Z_N = (z)_N + (\bar{z})_N - (\tilde{z})_N .$$

Note. The coefficients in the expansions $(\tilde{z})_N$ and z_N are functions of the order N of the approximation. This was not the case for the expansions obtained in Sections 2–4.

A Second Example

To illustrate the method of Vasil'eva, consider the nonhomogeneous, linear, constant coefficient system

$$dx/dt = a_1 x + a_2 y,$$
$$\epsilon(dy/dt) = b_1 x + b_2 y.$$

where b_2 is negative.

The corresponding reduced system

$$dx/dt = a_1 x + a_2 y,$$
$$0 = b_1 x + b_2 y$$

has $y = -(b_1/b_2)x$ as the only root of the equation $g(x\, y, t) \equiv b_1 x + b_2 y = 0$, so this root is *à fortiori* isolated. For the initial-value problem where $x(0)$ and $y(0)$ are prescribed for the full system and t is increasing, the boundary layer equation is

$$dy/d\tau = b_1 x^* + b_2 y,$$

where x^* has any arbitrary (fixed) value and $\tau = t/\epsilon$. Since $\partial g/\partial y \equiv b_2$ is negative, the root $y = -(b_1/b_2)x$ is stable. Furthermore, any initial point $(x(0), y(0), 0)$ will then lie in the domain of influence of this root.

Tikhonov's theorem, then, implies that the limiting solution, except for the y-limit at $t = 0$, will coincide with the solution of the reduced initial value problem, i.e.,

$$\left(x(0) \exp\left[\left(a_1 - \frac{b_1}{b_2} a_2\right) t\right], -\frac{b_1}{b_2} x(0) \exp\left[\left(a_1 - \frac{b_1 a_2}{b_2}\right) t\right]\right)$$

on any finite t interval. Indeed, if $y(0) \neq -(b_1/b_2)x(0)$, a boundary layer phenomenon (nonuniform convergence) must occur at $t = 0$.

The inner solution $z(\tau) = \sum_{k=0}^{\infty} z_k(\tau)\epsilon^k$ is constructed to satisfy

$$dx/d\tau = \epsilon(a_1 x + a_2 y),$$
$$dy/d\tau = b_1 x + b_2 y,$$

with $z(0)$ prescribed. Equating coefficients, we ask that

$$dx_k/d\tau = a_1 x_{k-1} + a_2 y_{k-1},$$
$$dy_k/d\tau = b_1 x_k + b_2 y_k,$$

for each $k \geqslant 0$ where $x_{-1} = y_{-1} \equiv 0$, $z_0(0) = z(0)$ and $z_k(0) = 0$ for $k > 0$. Integrating, then,

$$x_0(\tau) = x(0),$$

$$y_0(\tau) = \left(y(0) + \frac{b_1}{b_2} x(0)\right) e^{b_2 \tau} - \frac{b_1}{b_2} x(0),$$

$$x_1(\tau) = \left(a_1 - a_2 \frac{b_1}{b_2}\right) x(0)\, \tau + \frac{a_2}{b_2}\left(y(0) + \frac{b_1}{b_2} x(0)\right)(e^{b_2 \tau} - 1),$$

and

$$y_1(\tau) = -\frac{b_1}{b_2^2}\left(a_1 - \frac{a_2 b_1}{b_2}\right) x(0)(1 + b_2 \tau - e^{b_2 \tau})$$
$$+ \frac{b_1 a_2}{b_2^2}\left(y(0) + \frac{b_1}{b_2} x(0)\right)(1 + (b_2 \tau - 1)\, e^{b_2 \tau}).$$

Likewise, higher-order terms are uniquely obtainable by integrating successively, in order, for $x_k(\tau)$ and $y_k(\tau)$.

The outer solution

$$\bar{z}(t) = \sum_{k=0}^{\infty} \bar{z}_k(t)\, \epsilon^k$$

is constructed to formally satisfy the equations

$$d\bar{x}_k/dt = a_1\bar{x}_k + a_2\bar{y}_k ,$$
$$b_1\bar{x}_k + b_2\bar{y}_k = d\bar{y}_{k-1}/dt$$

for each integer $k \geqslant 0$ where $\bar{y}_{-1} \equiv 0$ and $\bar{x}_0(0) = x(0)$. Thus,

$$\bar{x}_0(t) = x(0) \exp\left[\left(a_1 - \frac{a_2b_1}{b_2}\right) t\right],$$

$$\bar{y}_0(t) = -\frac{b_1x(0)}{b_2} \exp\left[\left(a_1 - \frac{a_2b_1}{b_2}\right) t\right],$$

$$\bar{x}_1(t) = \left[\bar{x}_1(0) - \frac{a_2b_1}{b_2^2}\left(a_1 - \frac{a_2b_1}{b_2}\right) x(0)\, t\right] \exp\left[\left(a_1 - \frac{a_2b_1}{b_2}\right) t\right],$$

and

$$\bar{y}_1(t) = \left[-\frac{b_1}{b_2}\,\bar{x}_1(0) + \frac{a_2b_1^2}{b_2^3}\left(a_1 - \frac{a_2b_1}{b_2}\right) x(0)\, t \right.$$
$$\left. - \frac{b_1x(0)}{b_2^2}\left(a_1 - \frac{a_2b_1}{b_2}\right)\right] \exp\left[\left(a_1 - \frac{a_2b_1}{b_2}\right) t\right].$$

Higher-order terms may be analogously determined up to the selection of the initial values $\bar{x}_k(0)$.

Expanding the coefficients of the outer solution $\bar{z}(t)$ for small values of t, i.e.,

$$\bar{z}_k(t) = \sum_{l=0}^{\infty} z_{lk}t^l,$$

and inserting these expansions into $\bar{z}(t)$, we define $(\tilde{z})_N$ as the finite sum obtained by collecting all terms of order $\leqslant N$ in the two variables ϵ and t. Thus, for example,

$$\bar{x}_1(t) = \bar{x}_1(0) + \left[\bar{x}_1(0)\left(a_1 - \frac{a_2b_1}{b_2}\right) - \frac{a_2b_1}{b_2^2}\left(a_1 - \frac{a_2b_1}{b_2}\right) x(0)\right] t + \cdots$$

and

$$(\tilde{x})_0 = x(0),$$
$$(\tilde{y})_0 = -(b_1/b_2)\, x(0),$$
$$(\tilde{x})_1 = x(0) + tx(0)\left(a_1 - \frac{a_2b_1}{b_2}\right) + \epsilon\bar{x}_1(0),$$

and

$$(\tilde{y})_1 = -\frac{b_1 x(0)}{b_2} - t\,\frac{b_1}{b_2}\,x(0)\left(a_1 - \frac{a_2 b_1}{b_2}\right) - \frac{\epsilon b_1}{b_2}\,\bar{x}_1(0)$$

$$- \frac{\epsilon b_1 x(0)}{b_2^{\,2}}\left(a_1 - \frac{a_2 b_1}{b_2}\right).$$

Since b_2 is negative, $(\tilde{z})_0$ and $z_0(\tau)$ will have the same limit as $\tau \to \infty$. Further, $(\tilde{z})_1$ and $(z)_1 = z_0(\tau) + \epsilon z_1(\tau)$ will have the same limit as $\tau \to \infty$ (since $t = \tau\epsilon$) provided

$$\bar{x}_1(0) = -\frac{a_2}{b_2}\left(y(0) + \frac{b_1}{b_2}\,x(0)\right).$$

Note, however, that this is the value determined by formula (6.11) since, here,

$$f_0(\tau) = \left(a_1 - \frac{b_1 a_2}{b_2}\right) x(0) + a_2\left(y(0) + \frac{b_1}{b_2}\,x(0)\right) e^{b_2 \tau}$$

and

$$\tilde{f}_0(\tau) = \left(a_1 - \frac{b_1 a_2}{b_2}\right) x(0).$$

Higher-order initial values $\bar{x}_k(0)$ are successively obtained by matching $(\tilde{z})_k$ and $(z)_k$ as $\tau \to \infty$.

Introducing $(\bar{z})_N = \sum_{k=1}^{N} \bar{z}_k(t)\epsilon^k$, we obtain a uniformly valid solution to the initial-value problem on any finite interval $[0, T]$ by constructing

$$Z_N = (z)_N + (\bar{z})_N - (\tilde{z})_N\,,$$

i.e., for each $N \geqslant 0$, the solution z satisfies

$$z = Z_N + O(\epsilon^{N+1}) \qquad \text{uniformly as} \quad \epsilon \to 0.$$

Hence

$$X_0 = x(0)\exp\left[\left(a_1 - \frac{a_2 b_1}{b_2}\right) t\right],$$

$$Y_0 = -\frac{b_1}{b_2}\,x(0)\exp\left[\left(a_1 - \frac{a_2 b_1}{b_2}\right) t\right] + \left(y(0) + \frac{b_1}{b_2}\,x(0)\right) e^{b_2 t/\epsilon},$$

$$X_1 = \Big[x(0) - \epsilon\Big(\frac{a_2}{b_2}\left(y(0) + \frac{b_1}{b_2}\,x(0)\right)$$

$$+ \frac{a_2 b_1}{b_2^{\,2}}\left(a_1 - \frac{a_2 b_1}{b_2}\right) x(0)\,t\Big)\Big]\exp\left[\left(a_1 - \frac{a_2 b_1}{b_2}\right) t\right]$$

$$+ \epsilon\,\frac{a_2}{b_2}\left(y(0) + \frac{b_1}{b_2}\,x(0)\right) e^{b_2 t/\epsilon},$$

and

$$
\begin{aligned}
Y_1 = \Big[&- \frac{b_1}{b_2} x(0) + \epsilon \left(\frac{b_1 a_2}{b_2^2} \left(y(0) + \frac{b_1}{b_2} x(0) \right) \right. \\
&\left. + \frac{b_1 x(0)}{b_2^2} \left(a_1 - \frac{a_2 b_1}{b_2} \right) \left(a_2 \frac{b_1}{b_2} t - 1 \right) \right) \Big] \exp \left[\left(a_1 - \frac{a_2 b_1}{b_2} \right) t \right] \\
&+ \Big[\left(y(0) + \frac{b_1}{b_2} x(0) \right) \left(1 + \frac{a_2 b_1 t}{b_2} \right) \\
&+ \epsilon \left(\frac{b_1}{b_2^2} \left(a_1 - \frac{a_2 b_1}{b_2} \right) x(0) - \frac{b_1 a_2}{b_2^2} \left(y(0) + \frac{b_1}{b_2} x(0) \right) \right) \Big] e^{b_2 t/\epsilon}.
\end{aligned}
$$

Moreover these expansions may be verified, albeit with much labor, since such constant coefficient systems may be straightforwardly integrated.

7. Singular Perturbation Problems for Partial Differential Equations

In this section, we restrict attention to several representative problems. The literature abounds with many additional problems. In addition to the references cited below, the reader is urged to note, e.g., Birkhoff [3], Levin [64], Bobisud [4], Kisynski [59], and Smoller [97].

Examples where the Reduced Equation is of Lower Order

Levinson [66] considers the first boundary value problem for the linear second-order elliptic partial differential equation

$$\epsilon \Delta u + A(x, y)\, u_x + B(x, y)\, u_y + C(x, y)\, u = D(x, y) \tag{7.1}$$

where ϵ is a small positive parameter and Δ is the Laplacian operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$. Here, we will not present Levinson's results in their complete generality, but instead assume that the coefficients A, B, C, and D are sufficiently smooth in some open region R_0 of the x–y plane which contains a region R whose boundary S consists of a finite number of simple, closed curves. Further, let x and y be smooth functions of arc length s along each closed curve of S and let smooth boundary values u independent of ϵ be assigned along each such curve. Further, we ask that $A^2(x, y) + B^2(x, y) > 0$ in R_0 and that either R_0 be simply connected or that $C(x, y) < 0$ in R_0. Either hypothesis suffices to establish a maximum principle in $R \cup S$, i.e., a bound for $|u|$ on S determines a bound for $|u|$ in R. Under these hypotheses, the first boundary value problem for (7.1) will have a unique solution within $R \cup S$ for each $\epsilon > 0$.

As $\epsilon \to 0$, under quite general circumstances, we would expect the solutions of boundary value problems for (7.1) to converge to solutions of the reduced equation

$$A(x, y)\, u_x + B(x, y)\, u_y + C(x, y)\, u = D(x, y) \tag{7.2}$$

where the characteristic curves (or more simply, the characteristics) of this first-order equation satisfy

$$\frac{dx}{A(x, y)} = \frac{dy}{B(x, y)}.$$

The hypotheses listed above imply that this differential equation for the characteristics has no singular points in R_0.

Let us define the positive direction of the boundary curve S to be that direction on a tangent line to S from which a counterclockwise 90° rotation yields the interior normal, and let the arc length s of S be increasing in the positive direction. Let S_1 be a segment of one of the curves of S such that none of its tangents is a characteristic of the reduced equation (7.2), and let S_2 be that segment of a curve of S intersected by the characteristics of (7.2) passing through S_1, and conversely. Requiring that S_2 be nowhere characteristic,

$$B\, dx - A\, dy \neq 0$$

on both S_1 and S_2, so without loss of generality, let

$$B \frac{dx}{ds} - A \frac{dy}{ds} < 0 \quad \text{on} \quad S_1 \qquad \text{and} \qquad B \frac{dx}{ds} - A \frac{dy}{ds} > 0 \quad \text{on} \quad S_2 .$$

A closed simply-connected region in $R \cup S$ is formed by S_1 and S_2 and the two-characteristics of (7.2) joining the endpoints of S_1 and S_2. Hereafter, we refer to such a closed region as a "regular quadrilateral" Q.

Formal solutions of Eq. (7.1) can easily be constructed. For example, let $U(x, y, \epsilon)$ be an expansion of the form

$$U(x, y, \epsilon) = \sum_{k=0}^{\infty} U_k(x, y)\, \epsilon^k,$$

where U assumes the boundary values prescribed for u on S_1. Substituting into (7.1) and proceeding formally, we ask that the U_k satisfy

$$A(x, y)\, U_{kx}(x, y) + B(x, y)\, U_{ky} + C(x, y)\, U_k(x, y)$$
$$= \begin{cases} D(x, y) & \text{when} \quad k = 0, \\ -\Delta U_{k-1}(x, y) & \text{for} \quad k \geqslant 1, \end{cases}$$

with $U_0 = u$ and $U_k = 0$ for $k \geqslant 1$ on S_1. These linear equations may be uniquely solved successively since S_1 is a noncharacteristic curve for (7.2) and therefore for each of these equations.

In the event that the solution u of the boundary value problem for (7.1) converges as $\epsilon \to 0$ to the solution U_0 of the reduced boundary value problem for (7.2), a boundary layer phenomena can be expected along S_2 since the boundary values for U_0 on S_2 will not, in general, coincide with the boundary values prescribed there for u. Thus, we construct a formal boundary-layer-type solution of the homogeneous equation corresponding to (7.1) valid near S_2. Specifically, we set

$$z = \exp\left(-\frac{g(x, y)}{\epsilon}\right) h(x, y, \epsilon)$$

where $g = 0$ and $h = u - U$ on S_2 and g is positive away from S_2. Substituting z into the differential equation, we obtain

$$e^{-g/\epsilon}\Big[\frac{h}{\epsilon}(g_x^2 + g_y^2 - Ag_x - Bg_y)$$
$$+ ((A - 2g_x)\, h_x + (B - 2g_y)\, h_y + (C - \Delta g)\, h) + \epsilon\Delta h\Big] = 0.$$

Thus, we ask that g solve the nonlinear equation

$$g_x^2 + g_y^2 - Ag_x - Bg_y = 0.$$

Introducing the parameter t and setting $p = g_x$ and $q = g_y$, this equation is equivalent to the first-order system:

$$\frac{dx}{dt} = 2p - A,$$

$$\frac{dy}{dt} = 2q - B,$$

$$\frac{dg}{dt} = p(2p - A) + q(2q - B) = p^2 + q^2,$$

$$\frac{dp}{dt} = A_x p + B_x q,$$

and

$$\frac{dq}{dt} = A_y p + B_y q.$$

(The reader who is unfamiliar with the details of this procedure should see, e.g., Courant and Hilbert [21], pp. 75–84.) Letting $t = 0$ along S_2, the initial data is expressible as

$$x(s, 0) = x(s), \qquad y(s, 0) = y(s), \qquad g(s, 0) = 0$$

and since

$$\frac{dg}{ds} = p\frac{dx}{ds} + q\frac{dy}{ds} = 0 \qquad \text{and} \qquad dx^2 + dy^2 = ds^2$$

along S_2, the nontrivial determinations of p and q there are

$$p = -\left(B\frac{dx}{ds} - A\frac{dy}{ds}\right)\frac{dy}{ds}$$

and

$$q = \left(B\frac{dx}{ds} - A\frac{dy}{ds}\right)\frac{dx}{ds}.$$

Thus, $p(s, 0)$ and $q(s, 0)$ are completely specified. Moreover, since

$$\frac{dx}{ds}\frac{dy}{dt} - \frac{dy}{ds}\frac{dx}{dt} = B\frac{dx}{ds} - A\frac{dy}{ds} \neq 0$$

on S_2, g exists and is uniquely determined for this selection of p and q in a neighborhood of the noncharacteristic initial data curve S_2.

Knowing g, at least locally, we ask that $h(x, y, \epsilon)$ satisfy

$$(A - 2g_x)\,h_x + (B - 2g_y)\,h_y + (C - \Delta g)\,h = -\epsilon\Delta h.$$

Expanding, we let

$$h(x, y, \epsilon) = \sum_{k=0}^{\infty} h_k(x, y)\,\epsilon^k,$$

where each h_k satisfies the linear first-order equation

$$(A - 2g_x)\,h_{kx} + (B - 2g_y)\,h_{ky} + (C - \Delta g)\,h_k = -\Delta h_{k-1}$$

with

$$h_0 = u - U_0 \quad \text{on} \quad S_2,$$

while $h_k = -U_k$ for $k \geqslant 1$. For these equations, too, S_2 is noncharacteristic, so that the formal expansion for h can be uniquely obtained through any number of terms in the neighborhood of S_2 where g exists.

Using these formal results, Levinson proves the following.

Theorem 7.1. *Let $u(x, y)$ solve the first boundary value problem for*

$$\epsilon\Delta u + A(x, y)\, u_x + B(x, y)\, u_y + C(x, y)\, u = D(x, y)$$

in $R \cup S$ where a maximum principle holds. Then for each integer $N \geqslant 0$,

$$u(x, y) = \sum_{k=0}^{N} U_k(x, y)\, \epsilon^k + z_N(x, y, \epsilon) + O(\epsilon^{N+1/2})$$

uniformly in any regular quadrilateral Q where

$$z_N(x, y, \epsilon) = \begin{cases} e^{-g(x,y)/\epsilon} \sum_{k=0}^{N} h_k(x, y)\, \epsilon^k, & \textit{near } S_2 \\ e^{-\delta/\epsilon} \quad \textit{for some} \quad \delta > 0, & \textit{elsewhere in } Q \end{cases}$$

Note. (1) Eckhaus and de Jager [24] considerably extend Levinson's results and those of Visik and Lyusternik [107]. In particular, they obtain limited results at corner points and when the boundary S of the region R contains curves that are characteristics of the reduced equation.

(2) Oleinik [80] has obtained corresponding results for the second boundary value problem. Bobisud [6], [7] considers initial-boundary value problems for linear second-order parabolic equations with a small parameter multiplying the highest-order derivatives.

(4) In a series of papers, D. Huet (see, e.g., Huet [50]–[53], has established convergence theorems for singular perturbation problems for certain higher order elliptic partial differential equations in various Hilbert spaces. Weaker results for certain fourth-order equations have also been obtained by Davis [22].

Examples Illustrating Levinson's Theorem

Specializing, we consider the first boundary value problem for

$$\epsilon\Delta u + u_x + u_y = D(x, y) \tag{7.3}$$

in the unit square T with smooth boundary values

$$\begin{cases} u(0, y) = a(y), \\ u(1, y) = b(y), \\ u(x, 0) = c(x), \\ u(x, 1) = d(x), \end{cases} \tag{7.4}$$

prescribed on the boundary S of T. The characteristics of the reduced equation

$$u_x + u_y = D(x, y)$$

will, then, coincide with no tangent to S. Let S_1 and S_2 represent those edges of S along the lines $y = 1$ and $x = 0$, respectively. Then, we formally construct power series $U(x, y, \epsilon)$ and $H(x, y, \epsilon)$ such that an asymptotic solution $u(x, y)$ of the boundary value problem considered is given by

$$u(x, y) = U(x, y, \epsilon) + \exp\left(-\frac{g(x, y)}{\epsilon}\right) H(x, y, \epsilon)$$

where $y \geqslant x$, $U(x, 1, \epsilon) = d(x)$, $g(x, 1) = 0$, $g > 0$ elsewhere in T, and $H(0, y, \epsilon) = a(y) - U(0, y, \epsilon)$. Thus, we ask that U, g, and H satisfy the equations

$$U_x + U_y = D(x, y) - \epsilon \Delta U,$$

$$g_x^2 + g_y^2 - g_x - g_y = 0,$$

and

$$-2(H_x g_x + H_y g_y) + H_x + H_y - H \Delta g = -\epsilon \Delta H.$$

First, letting $U(x, y, \epsilon) = \sum_{k=0}^{\infty} U_k(x, y)\epsilon^k$, and proceeding formally, we obtain

$$U_0(x, y) = d(1 + x - y) + \int_1^y D(r - y + x, r)\, dr$$

and, successively, for $k \geqslant 1$,

$$U_k(x, y) = -\int_1^y \Delta U_{k-1}(r - y + x, r)\, dr,$$

where $\Delta U_j(r - y + x, r)$ represents $\Delta U_j(x, y)$ evaluated at $x = r - y + x$ and $y = r$.

Since $x = 0$ is not a characteristic of the nonlinear equation for g, its unique solution is $g = x$. Letting $H(x, y, \epsilon) = \sum_{k=0}^{\infty} h_k(x, y)\epsilon^k$, then, we ask that

$$-H_x + H_y = -\epsilon \Delta H$$

and obtain

$$h_0(x, y) = a(x + y) - U_0(0, x + y)$$

and, successively, for $k \geqslant 1$,

$$h_k(x, y) = -U_k(0, x + y) + \int_0^x \Delta h_{k-1}(t, x + y - t)\, dt.$$

Levinson's theorem, then, implies

Corollary 1. *Let $u(x, y)$ solve the first boundary value problem for*

$$\epsilon \Delta u + u_x + u_y = D(x, y)$$

in the unit square T. For each $N \geqslant 1$,

$$u(x, y) = \sum_{k=0}^{N} U_k(x, y)\, \epsilon^k + e^{-x/\epsilon} \sum_{k=0}^{N} h_k(x, y)\, \epsilon^k + O(\epsilon^{N+1/2})$$

uniformly in each regular quadrilateral

$$Q = \{(x, y) : (x, y) \in R \cup S,\quad 0 < \delta_1 \leqslant y - x \leqslant 1 - \delta_2 < 1\}$$

where the coefficients $U_k(x, y)$ and $h_k(x, y)$ are determined above.

Note. The analogous procedure can be used to construct the asymptotic solution in any regular quadrilateral

$$\tilde{Q} = \{(x, y) : (x, y) \in R \cup S,\quad 0 < \delta_1 \leqslant x - y \leqslant 1 - \delta_2 < 1\}$$

below the line $x = y$. Behavior along this line, except at $x = 0$ and $x = 1$, can also be obtained (see Eckhaus and de Jager [24]).

A second example is the equation

$$\epsilon \Delta u + u_x = D(x, y) \tag{7.5}$$

in the unit square T with smooth boundary values (7.4) prescribed. This problem was first considered, using different methods, in Wasow [110]. Here, the reduced equation,

$$u_x = D(x, y)$$

has the set of horizontal lines as characteristic curves. In particular, the boundary curves along $y = 0$ and $y = 1$ are characteristics.

Proceeding as usual, we obtain

Corollary 2. *Let $u(x, y)$ solve the equation*

$$\epsilon \Delta u + u_x = D(x, y)$$

in the unit square T with smooth boundary values

$$u(0, y) = a(y),$$
$$u(1, y) = b(y),$$
$$u(x, 0) = c(x), \qquad \text{and}$$
$$u(x, 1) = d(x).$$

For each $N \geqslant 0$, then,

$$u(x, y) = \sum_{k=0}^{N} U_k(x, y)\, \epsilon^k + e^{-x/\epsilon} \sum_{k=0}^{N} h_k(x, y)\, \epsilon^k + O(\epsilon^{\ \ +1/2})$$

uniformly in any regular quadrilateral

$$Q = \{(x, y)\colon\ 0 \leqslant x \leqslant 1,\ \ 0 < \delta_1 \leqslant y \leqslant 1 - \delta_2 < 1\}$$

where

$$U_0(x, y) = b(y) + \int_1^x D(s, y)\, ds,$$
$$h_0(x, y) = a(y) - U_0(0, y),$$

and, successively, for $k \geqslant 1$

$$U_k(x, y) = -\int_1^x \Delta U_{k-1}(s, y)\, ds$$

and

$$h_k(x, y) = -U_k(0, y) + \int_0^x \Delta h_{k-1}(s, y)\, ds.$$

Thus the asymptotic solution $u(x, y)$ is completely determined throughout the interior of T and along those portions of the boundary S which are not characteristics of the reduced equation. The complete asymptotic behavior along the lines $y = 0$ and $y = 1$ remains unknown.

An Extension of Levinson's Results to Two-Parameter Problems

In attempting to extend the methods outlined in Levinson [66], we obtain, for example,

Theorem 7.2. *Let $u(x, y)$ solve the first boundary value problem for*

$$\epsilon \Delta u - \mu A(x, y)\, u_x - C(x, y)\, u = 0 \tag{7.6}$$

in the unit square T where A and C are positive and ϵ and μ are small, positive, interrelated parameters. Further, let A, C, and the boundary values prescribed be sufficiently smooth and let $\epsilon/\mu^2 \to 0$ as $\mu \to 0$.

Then

$$u(x, y) \sim z_1\left(x, y, \frac{\epsilon}{\mu}, \frac{\epsilon}{\mu^2}\right) + z_2\left(x, y, \frac{\epsilon}{\mu}, \frac{\epsilon}{\mu^2}\right)$$

uniformly in any horizontal strip

$$Q : \{(x, y)\colon \ 0 \leqslant x \leqslant 1, \ \ 0 < \delta_1 \leqslant y \leqslant 1 - \delta_2 < 1\},$$

where

$$z_1\left(x, y, \frac{\epsilon}{\mu}, \frac{\epsilon}{\mu^2}\right) = h_1\left(x, y, \frac{\epsilon}{\mu}, \frac{\epsilon}{\mu^2}\right) \exp\left[-\frac{1}{\mu} g_1\left(x, y, \frac{\epsilon}{\mu^2}\right)\right]$$

and

$$z_2\left(x, y, \frac{\epsilon}{\mu}, \frac{\epsilon}{\mu^2}\right) = \begin{cases} h_2\left(x, y, \frac{\epsilon}{\mu}, \frac{\epsilon}{\mu^2}\right) \exp\left[-\frac{\mu}{\epsilon} g_2\left(x, y, \frac{\epsilon}{\mu^2}\right)\right] \\ \qquad\qquad \text{in a neighborhood of } \ x = 1 \\ O(e^{-\mu\delta/\epsilon}) \quad \text{for some } \ \delta > 0 \ \ \text{elsewhere in } Q. \end{cases}$$

The functions g_1 and g_2 are represented by power series in ϵ/μ^2, while h_1 and h_2 are represented by double power series in ϵ/μ and ϵ/μ^2. Their variable coefficients can be calculated successively by formal substitution into Eq. (7.6) *with the additional conditions*

$$h_1\left(0, y, \frac{\epsilon}{\mu}, \frac{\epsilon}{\mu^2}\right) = u(0, y), \quad g_1\left(0, y, \frac{\epsilon}{\mu^2}\right) = 0, \quad g_1 > 0$$

elsewhere in T,

$$h_2\left(1, y, \frac{\epsilon}{\mu}, \frac{\epsilon}{\mu^2}\right) = u(1, y), \quad g_2\left(1, y, \frac{\epsilon}{\mu^2}\right) = 0, \qquad \text{and} \qquad g_2 > 0$$

elsewhere,

Specifically, if we define z_{1N} and z_{2N} by terminating the expansions for h_1 and h_2, respectively, after all terms of order $\leqslant N$,

$$u(x, y) = z_{1N}\left(x, y, \frac{\epsilon}{\mu}, \frac{\epsilon}{\mu^2}\right) + z_{2N}\left(x, y, \frac{\epsilon}{\mu}, \frac{\epsilon}{\mu^2}\right) + O\left(\mu\left(\frac{\epsilon}{\mu^2}\right)^{N+1}\right)$$

uniformly in Q.

Note. (1) We do not attempt a boundary layer analysis in the neighborhoods of the lines $y = 0$ and $y = 1$ since they are characteristics of the "semi-reduced" or "intermediate" equation

$$\mu A(x, y)\, u_x - C(x, y)\, u = 0.$$

(2) General results for equations of the form

$$\epsilon(\Delta u + a(x, y)\, u_x + b(x, y)\, u_y + c(x, y)\, u)$$
$$+ \mu(e(x, y)\, u_x + f(x, y)\, u_y + g(x, y)\, u) + h(x, y)\, u = r(x, y)$$

for h negative and ϵ and μ small, positive, interrelated parameters simultaneously approaching zero are reported in O'Malley [84] for the Dirichlet problem and O'Malley [87] for the Robin problem.

An Example where the Reduced Equation is of a Different Type

Latta [63] considers the first boundary value problem for the equation

$$\epsilon u_{yy} + u_{xx} - u_y = 0 \tag{7.7}$$

in fairly general domains of the x–y plane. Note that this differential equation is elliptic for $\epsilon > 0$ and its first boundary value problem then admits a unique solution, but the reduced equation

$$u_{xx} - u_y = 0$$

is parabolic. Restricting attention to the unit square T, let smooth boundary values be prescribed. In the event that the solution of the boundary value problem converges to a solution of the reduced equation, we can anticipate boundary layer behavior in the vicinity of the upper edge of T since the solution of the reduced equation is uniquely determined throughout T by the boundary values assumed on the other three edges.

Thus, we attempt to determine an asymptotic solution $u(x, y, \epsilon)$ of the form

$$u(x, y, \epsilon) = A(x, y, \epsilon) + B(x, y, \epsilon) \exp\left(-\frac{g(x, y)}{\epsilon}\right)$$

where A and B represent power series in ϵ with variable coefficients and $g(x, y)$ has the value zero along the line $y = 1$, but is positive elsewhere in T. Formally substituting this expression into (7.7), we have

$$\begin{aligned} &\epsilon A_{yy} + A_{xx} - A_y \\ &\quad + \frac{\exp(-g(x, y)/\epsilon)}{\epsilon^2} [Bg_x^2 + \epsilon(-2B_x g_x - Bg_{xx} + Bg_y^2 + Bg_y) \\ &\quad + \epsilon^2(B_{xx} - 2B_y g_y - Bg_{yy} - B_y) + \epsilon^3 B_{yy}] = 0. \end{aligned}$$

Setting $A(x, y, \epsilon) = \sum_{k=0}^{\infty} a_k(x, y)\epsilon^k$, then, we ask that the coefficients a_k satisfy

$$a_{kxx} - a_{ky} = -a_{k-1,yy}$$

where $a_{-1} \equiv 0$ and the boundary values $a_0(x, y) = u(x, y)$ and $a_k(x, y) = 0$ for $k \geqslant 1$ are prescribed on the lower base and the vertical sides of T. (This assumes that bounded derivatives $a_{k-1,yy}$ exist throughout T.) Thus, the a_k's can be uniquely determined successively, being solutions of well-posed boundary value problems for nonhomogeneous parabolic equations.

Likewise, asking that

$$Bg_x^2 = 0$$

for nonzero B, we set $g_x = 0$. Then setting

$$-2B_x g_x - Bg_{xx} + Bg_y^2 + Bg_y = Bg_y(g_y + 1) = 0,$$

with $g(x, 1) = 0$, we obtain

$$g(x, y) = 1 - y.$$

Proceeding, we set

$$B_{xx} - 2B_y g_y - Bg_{yy} - B_y + \epsilon B_{yy} = B_{xx} + \epsilon B_{yy} + B_y = 0$$

where

$$B(x, y, \epsilon) = \sum_{k=0}^{\infty} b_k(x, y)\, \epsilon^k.$$

The coefficients b_k can be uniquely obtained by solving the following well-posed boundary value problems successively:

$$b_{0xx} + b_{0y} = 0$$

with

$$b_0(x, 1) = u(x, 1) - a_0(x, 1)$$

and

$$b_0(x, y) = 0 \quad \text{on the vertical sides of } T$$

and for $k \geqslant 1$, assuming that $b_{k-1,yy}$ exists and is bounded throughout T,

$$b_{kxx} + b_{ky} = -b_{k-1,yy}$$

with

$$b_k(x, 1) = -a_k(x, 1)$$

and

$$b_k(x, y) = 0 \quad \text{on the vertical sides of } T.$$

Applying an appropriate maximum principle argument, Latta is able to prove:

Theorem 7.3. *Let $u(x, y)$ solve the first boundary value problem for the equation*

$$\epsilon u_{yy} + u_{xx} - u_y = 0$$

in the unit square T. The, for each $N \geqslant 1$,

$$u(x, y) = \sum_{k=0}^{N} a_k(x, y)\, \epsilon^k + \left(\sum_{k=0}^{N} b_k(x, y)\, \epsilon^k\right) \exp\left(-\frac{(1-y)}{\epsilon}\right) + O(\epsilon^{N+1})$$

as $\epsilon \to 0$ uniformly in T where $a_k(x, y)$ and $b_k(x, y)$ are the coefficients determined above.

Note. (1) In Latta [63], the error term for the general problems considered is proved to be $O(\epsilon^{N+1-\delta})$ for any $\delta > 0$. Later, in unpublished work, Latta has shown that the error is, in fact, $O(\epsilon^{N+1})$ for the problem considered here.

(2) Zlamal [119] considers more general parabolic equations as the limiting case of certain elliptic equations. He carefully analyzes difficulties arising because the existence of bounded derivatives of the formally constructed asymptotic solution is not assured.

(3) Long ago, Hadamard [40] showed how to obtain the fundamental solution of the heat equation as the limit of the Riemann function for a hyperbolic equation involving a small parameter in such a manner that the heat equation is its limiting form. Likewise, Zlamal [120] (and certain previous papers) treats more general parabolic equations as the limit of certain hyperbolic equations, using the Fourier transform method to show asymptotic convergence. Bobisud [5] considers analogous problems for systems of partial differential equations of the first order in the "time" variable t. Recently, several authors (see, e.g., Lions [70], [71], and Kohn and Nirenberg [60]) have utilized results for singularly perturbed elliptic equations as a smoothing technique. This method is known as elliptic regularization.

A Problem for Oseen Flow

Latta [63] also considers Oseen flow for a viscous incompressible fluid past a semi-infinite flat plate at zero angle of attack. Letting the plate coincide with the positive x-axis, $0 \leqslant x < \infty$, in the x–y plane, the equations of motion are

$$\begin{cases} u_x + p_x = \nu \Delta u, \\ v_x + p_y = \nu \Delta v, \\ u_x + v_y = 0, \end{cases} \tag{7.8}$$

where ν is the coefficient of viscosity, assumed to be small and positive, and u and v represent the perturbation components of the flow velocity. (In these equations, the constant free-stream velocity, or, equivalently, the flow velocity at upstream infinity, and the density are both normalized to have the value one.) The boundary conditions are that

$$u = u_0 \quad \text{and} \quad v = 0 \quad \text{along the plate,}$$

while

$$u = v = p = 0 \quad \text{at upstream infinity.}$$

Thus, u_0 represents the relative velocity of the plate with respect to the constant free stream velocity. For a fixed plate, then, $u_0 = -1$.

Making the change to parabolic coordinates (the "optimal coordinates," as later defined in Kaplan [54]), we set $z = x + iy = w^2 = (\xi + i\eta)^2$, so that the z-plane corresponds to $\eta \geqslant 0$ and the plate lies along the

ξ-axis, $\eta = 0$. In (ξ, η) coordinates, then, the equations of motion are rewritten:

$$\nu \Delta u - 2\xi u_\xi + 2\eta u_\eta - 2\xi p_\xi + 2\eta p_\eta = 0,$$
$$\nu \Delta v - 2\xi v_\xi + 2\eta v_\eta - 2\xi p_\eta - 2\eta p_\xi = 0,$$
$$\xi u_\xi - \eta u_\eta + \xi v_\eta + \eta v_\xi = 0.$$

The classical boundary layer analysis (see, e.g., Goldstein [37]) predicts on physical grounds the boundary layer thickness to be of order $\sqrt{\nu}$. Introducing ξ and $\sigma = \eta/\sqrt{\nu}$, then, as the independent variables in the boundary layer, the resulting boundary layer equations have the asymptotic solutions

$$u \sim u_0 \operatorname{erfc} \sigma,$$
$$v \sim 0,$$
$$p \sim 0,$$

as $\nu \to 0$ where

$$\operatorname{erfc} \sigma = \frac{2}{\sqrt{\pi}} \int_\sigma^\infty e^{-t^2}\, dt.$$

Thus, in an attempt to obtain a uniformly valid asymptotic solution of the equations of motion for all ξ and η, we introduce the boundary layer stretching coordinate $g(\xi, \eta)/\nu$ where g is zero along the plate, i.e., when $\eta = 0$, but g is positive elsewhere. Then, we formally set

$$u = u_1(\xi, \eta) \operatorname{erfc} \sqrt{\frac{g(\xi, \eta)}{\nu}},$$
$$+ \sqrt{\nu} \left(u_2(\xi, \eta, \sqrt{\nu}) + u_3(\xi, \eta, \sqrt{\nu}) \exp\left(- \frac{g(\xi, \eta)}{\nu} \right) \right),$$
$$v = \sqrt{\nu} \left(v_2(\xi, \eta, \sqrt{\nu}) + v_3(\xi, \eta, \sqrt{\nu}) \exp\left(- \frac{g(\xi, \eta)}{\nu} \right) \right),$$

and

$$p = \sqrt{\nu} \left(p_2(\xi, \eta, \sqrt{\nu}) + p_3(\xi, \eta, \sqrt{\nu}) \exp\left(- \frac{g(\xi, \eta)}{\nu} \right) \right),$$

where, as usual, u_2, u_3, v_2, v_3, p_2 and p_3 represent power series in $\sqrt{\nu}$ with variable coefficients. Note that $\operatorname{erfc} \sqrt{g(\xi, \eta)/\nu}$ and $\exp(-g(\xi, \eta)/\nu)$ are both exponentially small as $\nu \to 0$ away from the boundary $\eta = 0$.

Proceeding formally and applying the boundary conditions, we obtain

$$g(\xi, \eta) = \eta^2$$

and the solutions

$$u = u_0 \operatorname{erfc} \frac{\eta}{\sqrt{\nu}} + \sqrt{\nu} \left(\frac{u_0 \eta}{\sqrt{\pi} (\xi^2 + \eta^2)} \right) (1 - e^{-\eta^2/\nu}),$$

$$v = - \sqrt{\nu} \left(\frac{u_0 \xi}{\sqrt{\pi} (\xi^2 + \eta^2)} \right) (1 - e^{-\eta^2/\nu}),$$

$$p = - \sqrt{\nu} \frac{u_0 \eta}{\sqrt{\pi} (\xi^2 + \eta^2)} .$$

These formally determined solutions are exact for all values of ν, so, *à fortiori*, are also the asymptotic solutions desired as $\nu \to 0$.

Note. (1) This solution was also obtained in Lewis and Carrier [68] using Wiener–Hopf techniques.

(2) The form of solution suggests that one could have proceeded, without the benefit of physical reasoning, by formally expanding u, v, and p in the form

$$\sum_{k=1}^{\infty} f(\xi, \eta, \zeta) \, \nu^{k/2}$$

where $\zeta = g(\xi, \eta)/\nu$ is treated as a new independent variable, in the manner of Cochran [14] and O'Malley [85].

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